RADIATION FROM AN ANTENNA ENTERING THE MARTIAN ATMOSPHERE

Thesis by

John D. Norgard

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1969

(Submitted May 15, 1969)

Copyright © by

JOHN DENNIS NORGARD

1970

ACKNOWLEDGEMENTS

The author is indebted to his research advisor, Professor Charles H. Papas, for his continued guidance and encouragement during the course of this work.

Also acknowledged, with thanks, are the efforts of Mrs. Ruth Stratton in preparing the final manuscript.

ABSTRACT

The interaction between the ionized wake of a capsule entering the Martian atmosphere and the circularly polarized radiation emitted by an antenna located on the aft part of the capsule is theoretically investigated in this study. A simplified mathematical model of the atmosphere of Mars, the entry-trajectory of the capsule, and the flow field surrounding the capsule are used in the analysis. The near wake of the capsule is approximated by a cylindrically stratified plasma shell consisting of n plasma regions. The plasma in each region is assumed to be homogeneous, anisotropic, and conducting, and moving at a uniform velocity relative to the antenna. The antenna is represented by a turnstile antenna located off-axis $\lambda_o/4$ above an infinite ground plane and operates at the signal frequencies of 400 MHz and 2.295 GHz.

Integral expressions for the cylindrical components of the field vectors are obtained by a rigorous relativistic formulation of the problem, and are evaluated using the techniques of asymptotic expansions to yield the radiation patterns of the antenna. Radiation patterns for the special case of an on-axis antenna radiating through a uniform, lossless, and isotropic plasma shell are presented and are compared with the free space patterns.

The analysis shows that blackout occurs during the entry of a capsule into the Martian atmosphere. Before and after blackout, the radiation patterns of the antenna exhibit an on-axis null region whose angular extent is proportional to the electron concentration of the plasma. Also, sharp peaks which are attributed to leaky wave radiation, are present in the null region of the patterns. For the non-null

region of the patterns, the values of the gain function of the antenna oscillate about the free space values. As the electron concentration of the plasma increases, the peaks in the radiation patterns become more numerous and more sharply defined. The effects of the motion of the plasma on the radiation emitted by the antenna are to shift the peaks of the radiation patterns to smaller cone angles and to introduce more peaks into the patterns.

For the low velocity case corresponding to an entry into the Martian atmosphere, no serious motional or depolarization effects occur, and communications with the capsule can be satisfactorily carried out when the condition of blackout does not exist.

TABLE OF CONTENTS

I. INTRODUCTION

A. The Communication Blackout Problem

In this age of space exploration it has become necessary to analyze the problem of communicating with an instrumented space probe as it encounters an extraterrestrial planet. Recently, certain preliminary aspects of an exploratory mission to an extraterrestrial planet have been studied and have been shown to be scientifically meaningful and technically feasible.

The objectives of an exploratory mission to an extraterrestrial planet are

(1) To perform an atmospheric entry and obtain diagnostic data on the physical properties of the planetary atmosphere;

(2) To survive a landing and return desirable scientific data on the properties of the atmosphere at the surface of the planet;

(3) To perform flyby scientific experiments that complement the entry mission.

The underlying objective of this study is to analyze the electromagnetic aspects of communicating with an instrumented space probe as it encounters the atmosphere of an extraterrestrial planet and attempts a soft-landing on the surface of the planet. Such an undertaking requires an approximate knowledge of the properties of the atmosphere at the surface of the planet and the variation of these properties with altitude above the surface of the planet. This study describes an experiment designed to approximately determine this knowledge. This experiment can be performed on a flyby mission before the entry mission takes place. The trajectory of the space probe on the flyby mission is developed so that the probe will disappear behind the limb of the planet as seen by the Earth and will be occulted by the planet. Before its intercept by the surface of the planet, and again after its reappearance on the opposite side of the planet, the s-band telemetry signal of the probe will transverse the atmosphere of the planet. The refractive properties of the atmosphere will cause changes to appear in the phase, frequency, and amplitude of the signal received at the Earth. It is the measurement of these changes that will form the basis for a determination of the properties of the atmosphere of the planet.

The data obtained from the occultation experiment can then be used to design a survivable landing capsule. In particular, the scale height of the atmosphere must be accurately known so that the capsule can be designed to withstand the aerodynamic heating and deceleration forces that will be encountered during the entry; and the surface pressure and density of the atmosphere must be accurately known for the proper design of the descent parachutes. Other properties of interest in this study are the composition of the atmosphere, the surface temperature, the adiabatic lapse rate and the specific heat of the gases constituting the atmosphere, and the tropopause altitude.

After a survivable landing capsule is designed, the problem still remains of receiving the data telemetered to Earth. Certain difficulties can arise that will now be discussed.

As the capsule encounters the atmosphere of the planet, a resistance to its motion by the atmospheric gases will result in an exchange of energy between the capsule and the gases, and a subsequent ionization of the gases. The degree of ionization depends on the composition and physical properties of the atmosphere, as well as on the entry-trajectory characteristics of the capsule. If the degree of ionization is substantial, then a shock-induced envelope of ionized gases will form around, and trail, the capsule as it passes through the atmosphere of the planet (Figure 1).

The shock-induced envelope of ionized gases constitutes a moving plasma flow field, i.e., a macroscopically neutral ionized gas consisting primarily of free electrons, free ions, and neutral particles, that moves with some velocity relative to the capsule. The plasma can interact with the electromagnetic radiation received by and transmitted from the capsule, and communications to and from the capsule can be seriously disrupted. If this happens, changes, the severity of which will depend largely on the degree of ionization and collision frequency of the gases constituting the plasma, can be expected in the phase, frequency and amplitude of the s-band telemetry signal received at the Earth. At hypersonic entry velocities the combined effects of phase increase, doppler shift, and refractive defocusing attenuation can be sufficient to cause "blackout", i.e., complete loss of signal, even at extremely high frequencies. It is desirable to establish whether or not blackout will occur during the entry of the capsule into the atmosphere of the planet, so that data collected during blackout

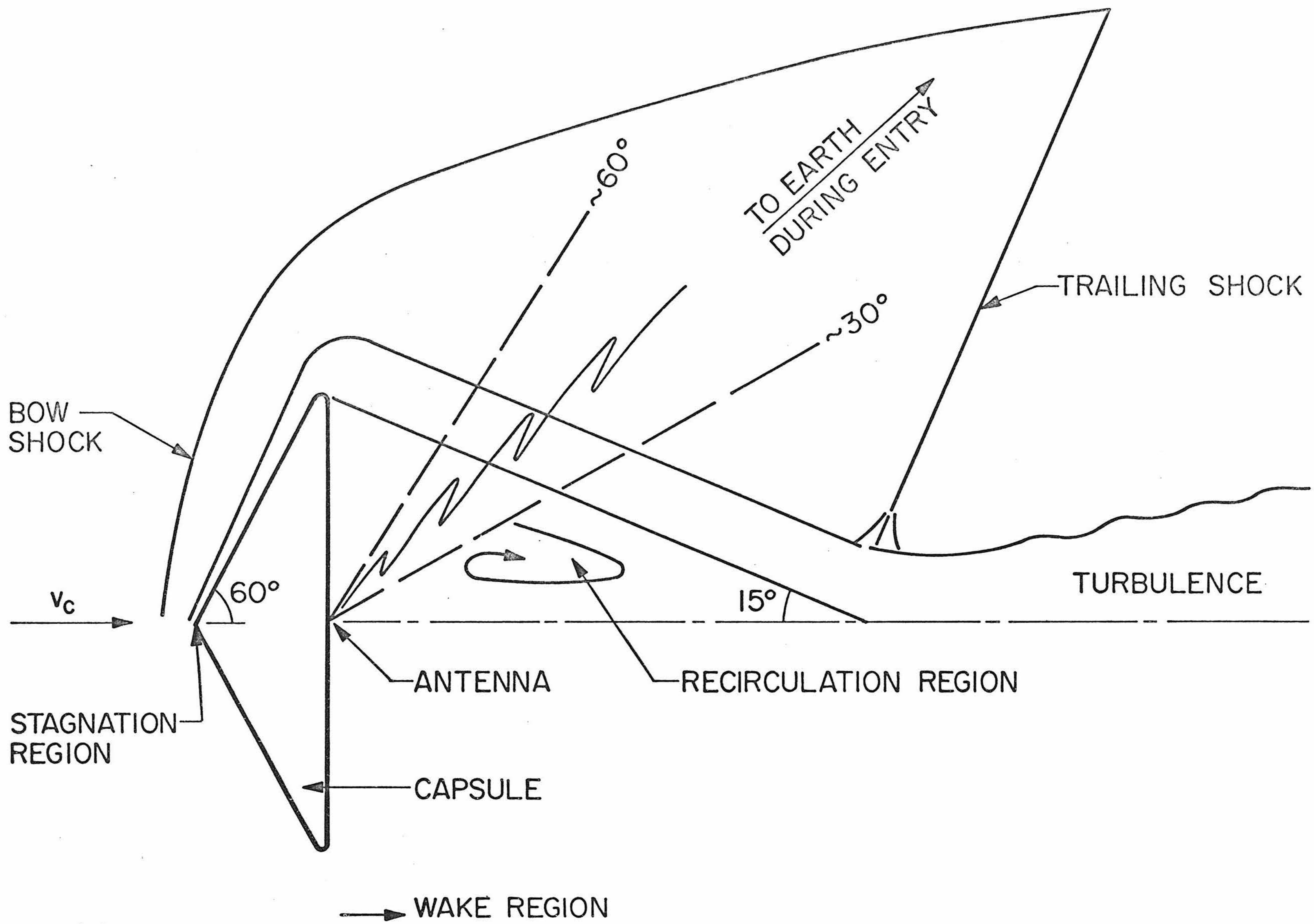


Fig. 1 Plasma Flow Field

can be stored and replayed after the end of blackout.

B. Previous Studies

Although numerous detailed studies have recently been conducted on the communication blackout problem, most of them have been concerned with the Earth reentry problem. In these cases, propagation takes place in the forward direction through a plasma sheath, since for Earth reentry the antennas are usually located on the forward part of the capsule. The entry of a capsule into the atmosphere of an extraterrestrial planet poses a different set of problems. In such cases, propagation takes place through the near and far wakes behind the capsule, since the antennas are usually mounted on the aft part of the capsule. Therefore, a description of the plasma is required in the wake region. It will be sufficient to determine only the electron concentration and collision frequency profiles of the plasma in the wake region for the work that follows. Determining these properties of the plasma is a difficult problem in itself, since it involves a detailed knowledge of the thermodynamic properties and chemical composition of a multicomponent, high temperature, nonequilibrium gas flow field, and knowledge of the atmosphere of the planet and the entry-trajectory of the capsule. Certain reasonable approximations will be introduced to surmount these thermodynamic and chemical problems.

C. Mars

Although many of the aspects of this study are applicable to any extraterrestrial planet, the specific results and examples are limited to the Martian atmosphere. This choice was made because of the great amount of interest accorded Mars at the time of writing, and also, because of the availability of astronomical and occultation data on the planet. The electromagnetic work that follows will be done in complete generality, however.

One of the present designs conceived for the Mariner Mars mission is such that the space probe will acquire an orbit trajectory around Mars. A capsule will then be released on an impact trajectory to enter the Martian atmosphere and, at a predetermined altitude, deploy a parachute and extract a lander which will soft-land on the surface of the planet. This design uses transmission frequencies of 400 MHz and 2.295 GHz, the proposed transmission frequencies for communication links between the landing capsule and a relay bus (space probe) and directly between the landing capsule and the Earth, respectively. This study will also be limited to these two communication frequencies.

D. Outline

This study naturally divides into two main problems: a thermodynamic-chemical problem, and an electromagnetic problem. The thermodynamic-chemical problem further divides conveniently into three parts. (1) given the available astronomical and occultation data on certain key properties of the Martian atmosphere, one can construct a reasonable model of the entire atmosphere; (2) given a description of the landing capsule and the initial entry-trajectory data, one can construct the entry-trajectory of the capsule into the Martian atmosphere. This calculation will, of course, depend on the model atmosphere developed in part (1); (3) given the model atmosphere and entry-trajectory developed in parts (1) and (2), one can construct the electron concentration and the collision frequency profiles of the moving plasma flow field in the wake region of the capsule as a function of altitude above the surface of Mars.

After a detailed knowledge of the properties of the plasma is developed, the analysis of the electromagnetic interaction of the moving plasma flow field with the s-band telemetry signal will be undertaken. The following sections of this study will consist of a detailed analysis of the thermodynamic-chemical and the electromagnetic problems using the planet Mars as a specific example.

II. THE CHEMICAL-THERMODYNAMIC PROBLEM

A. The Martian Atmosphere

1. Introduction

Certain key properties of the Martian atmosphere, such as surface pressure, density, temperature, are used in this section to construct a reasonable model of the entire atmosphere.

First, the occultation experiment used to develop this information is described. Then the results of the occultation experiment are presented from these data and various models of the Martian atmosphere are proposed. Since there is some uncertainty about the basic reliability of the occultation data, the actual properties of the Martian atmosphere and the variation of these properties with altitude above the surface of Mars are still not precisely known. Consequently, there is some uncertainty about which model derived from the occultation data most accurately describes the Martian atmosphere. Therefore, two model atmospheres, VM-4 and VM-8, that have been chosen to represent a reasonable range of expected conditions, are used throughout the remainder of the study.

2. The Occultation Experiment

Despite the great amount of attention accorded its study in the past few years, the structure of the Martian atmosphere has remained, until recently, an unsolved problem subject to many diverse interpretations. Previous knowledge of such atmospheric properties as surface pressure, density, temperature, and scale height was quite inexact. For example, the surface pressure as deduced from spectroscopic observations was thought to be between 10 and 40 mb., instead of the 85 mb. figure previously derived from Rayleigh scattering measurements (1). The vertical structure of the atmosphere, including properties of the tropopause and scale height of the stratosphere, being inaccessible to direct Earth based measurements, was therefore only estimated from assumptions about the atmospheric constituents and temperature. Also, the properties of the ionosphere were open only to the postulation of models based in turn on the estimated structure of the upper atmosphere.

The s-band telemetry occultation experiment (2) on the Mariner Mars 1965 flyby--the design of which will now be discussed--offered the earliest opportunity to significantly reduce the uncertainty surrounding the Martian atmosphere. The geometry of the occultation experiment is shown schematically in Figure 2. At entry into occultation the space probe was 25, 570 km distant from the limb of the planet and traveling at a velocity of 2.07 km/s normal to the Earth-Mars line. The point of tangency on the surface of Mars was located at latitude 50.5° S and longitude 177° E, corresponding to a point between Electris and Mare Chronium. At the time of exit from occultation, the distance from the limb of the planet had increased to 39, 130 km, and the point of tangency, which fell within Mare Acidalium on the surface of Mars, was located at latitude 60° N and longitude 34° W. As the space probe passed behind the limb of Mars and emerged from the opposite limb 54 minutes later, the path of its s-band telemetry signal passed through the Martian atmosphere. The presence of the atmosphere caused the velocity of propagation of the signal to deviate from the velocity in free space because of the nonunity index of refraction of the neutral

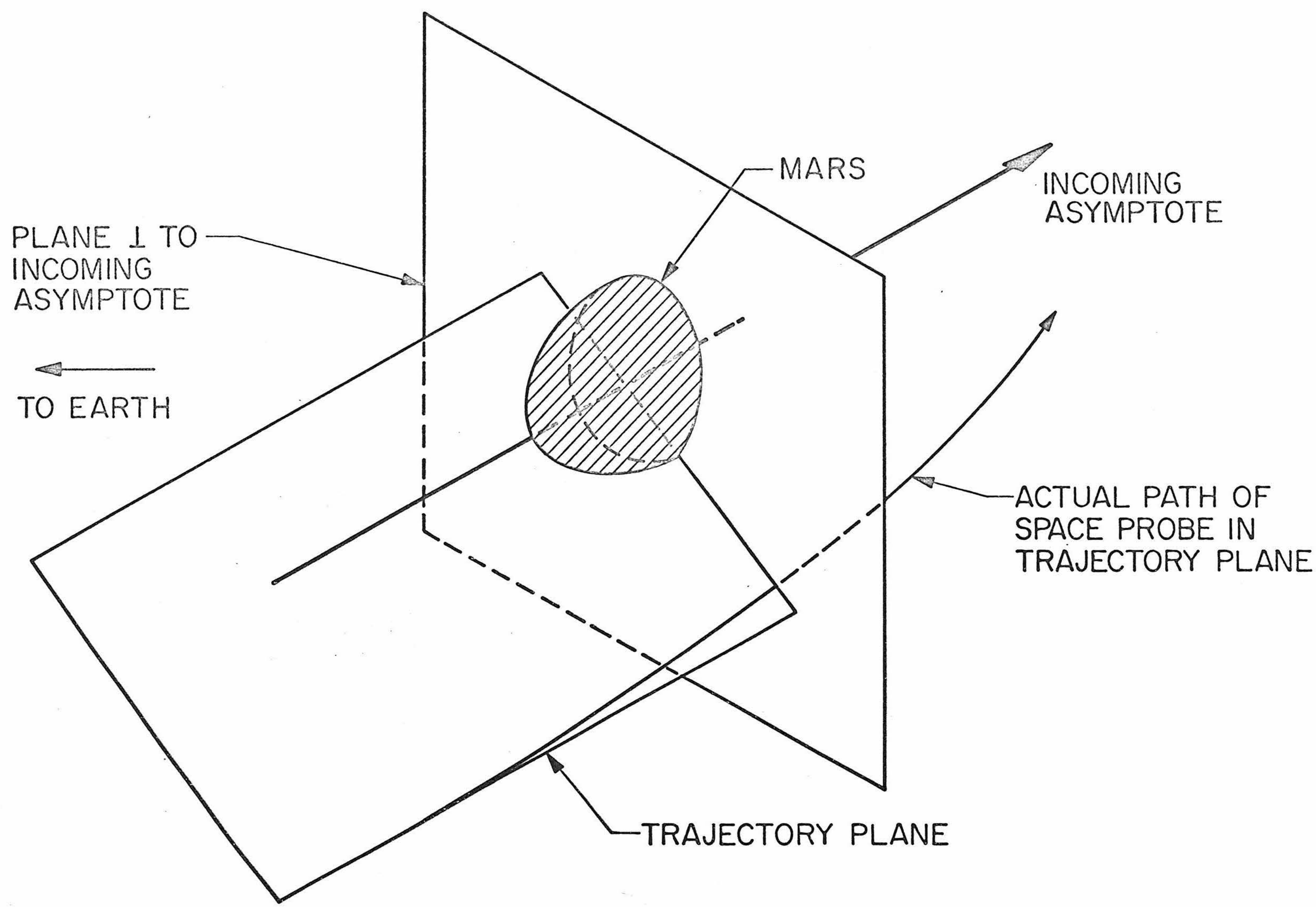


Fig. 2. Occultation Geometry

and ionized layers of atmospheric media. Also, the radial gradient of the effective index of refraction of the atmosphere caused the signal to be refracted from a straight-line path. Therefore changes in the apparent phase path length between the space probe on its trajectory and the tracking antenna on the Earth were observed. Since the trajectory of the space probe was precisely determined from pre- and post-encounter tracking, the effects of the atmosphere were apparent when the actual phase path length of the signal was compared with the phase path length predicted by the orbit of the space probe. Also, since the amount of phase deviation of the signal changed with time, the received frequency of the signal differed from the predicted value. Too, the lens-like effect of the refractivity gradient in the atmosphere that caused the signal to diverge led to a reduction of the received signal power. Thus changes in the phase, frequency and amplitude of the signal received on the Earth were observed and recorded; and this information constituted the raw data of the experiment.

Figures 3 and 4 show the (observed minus predicted) doppler shifts and phase differences based on data received at the various DSIF (Deep Space Instrumentation Facilities) stations. Both closed loop data (data taken with the transmitting frequency reference of the space probe provided by a frequency standard on the Earth) and open loop data (data taken with the transmitting frequency reference provided by an on-board crystal oscillator) were recorded. In the later mode, the precision of the phase measurements is significantly degraded. One may observe that the data from the various sources show a high degree of consistency. The most important results of the occultation experiment

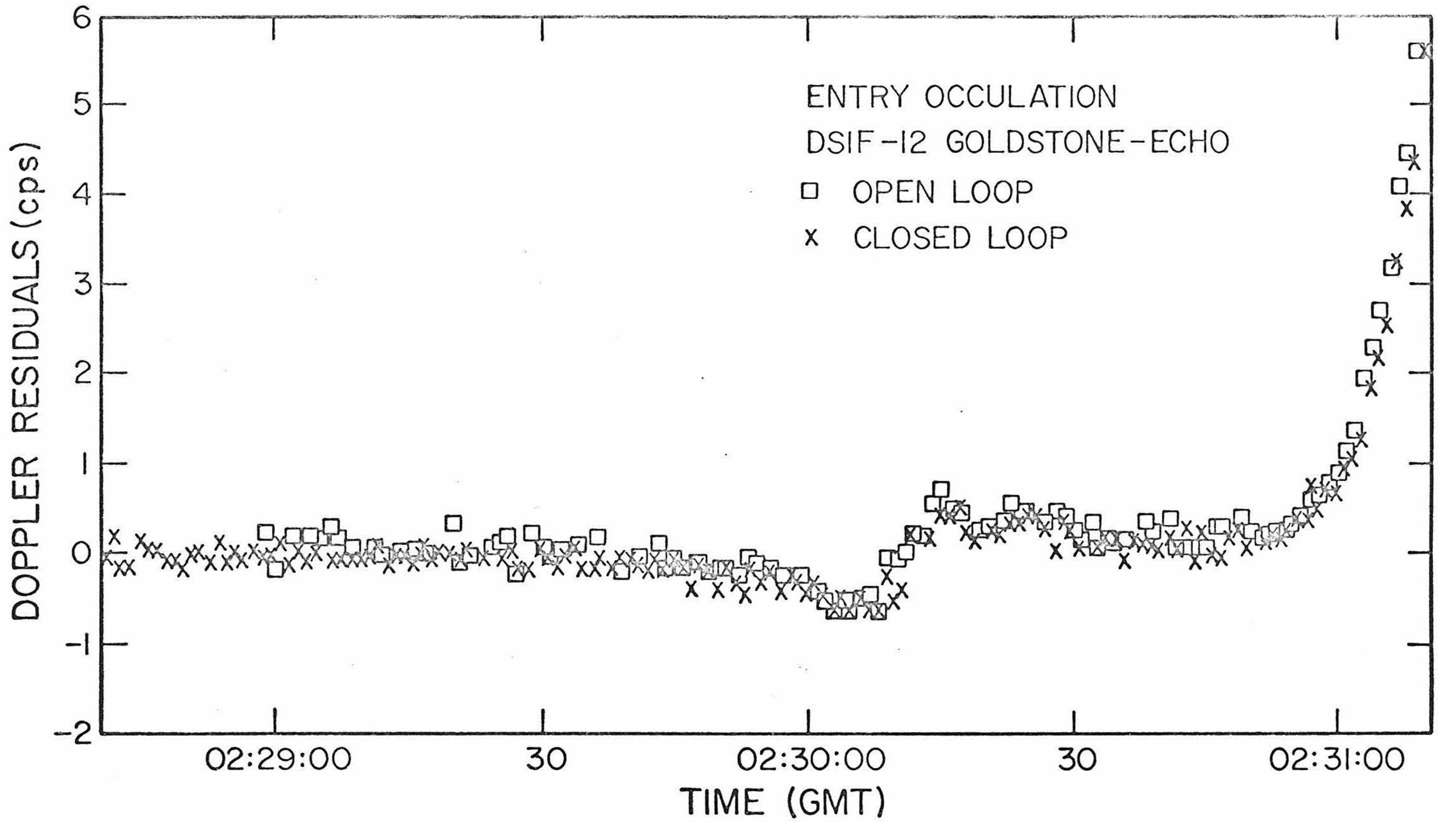


Fig. 3. Doppler Shift (Entry Residuals)

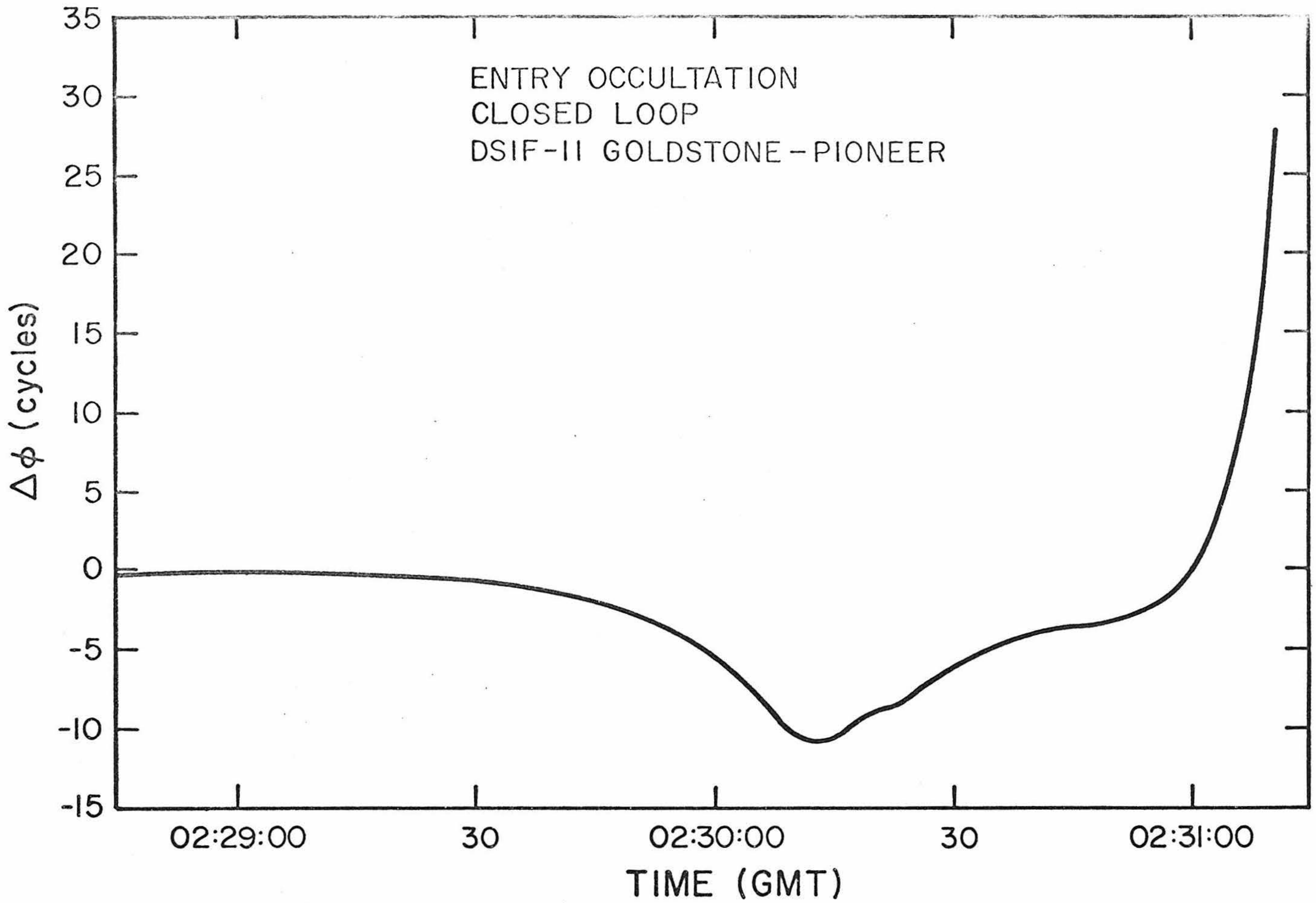


Fig. 4. Phase Difference (Entry Residual Sum)

were the determination of the properties of the Martian atmosphere at the surface of the planet from the occultation data. Actually, the occultation data indirectly gave the values of the properties of the atmosphere at the place on Mars that occulted the telemetry signal. There was the disturbing possibility that the place might have been, however, a high mountain peak. In analogy to the Earth, this would cause the values of the derived properties of the atmosphere to be off perhaps by half those values found at most points on the surface of Mars. To mitigate this problem, it was important to observe both the space probe immersion into and emersion out of occultation and compare the data taken in each instance.

Figure 5 shows the (observed minus predicted) doppler shifts based on the data received during the space probe emersion. A comparison of the entry and exit doppler shifts shows that both sets of data are similar. Therefore, there was confidence that the data were valid and that the data were indirect measurements of the properties of the Martian atmosphere near the nominal surface of the planet.

Analyses of the changes in phase, frequency, and amplitude of the telemetry signal were used to infer some of the properties of the Martian atmosphere. The properties of interest in this study are

(1) The composition of the atmosphere,
(2) The pressure, density, and temperature at the surface of the planet,
(3) The specific heat of the gases constituting the atmosphere,
(4) The adiabatic lapse rate of the stratosphere,
(5) The scale height of the tropopause,
(6) The altitude of the tropopause.

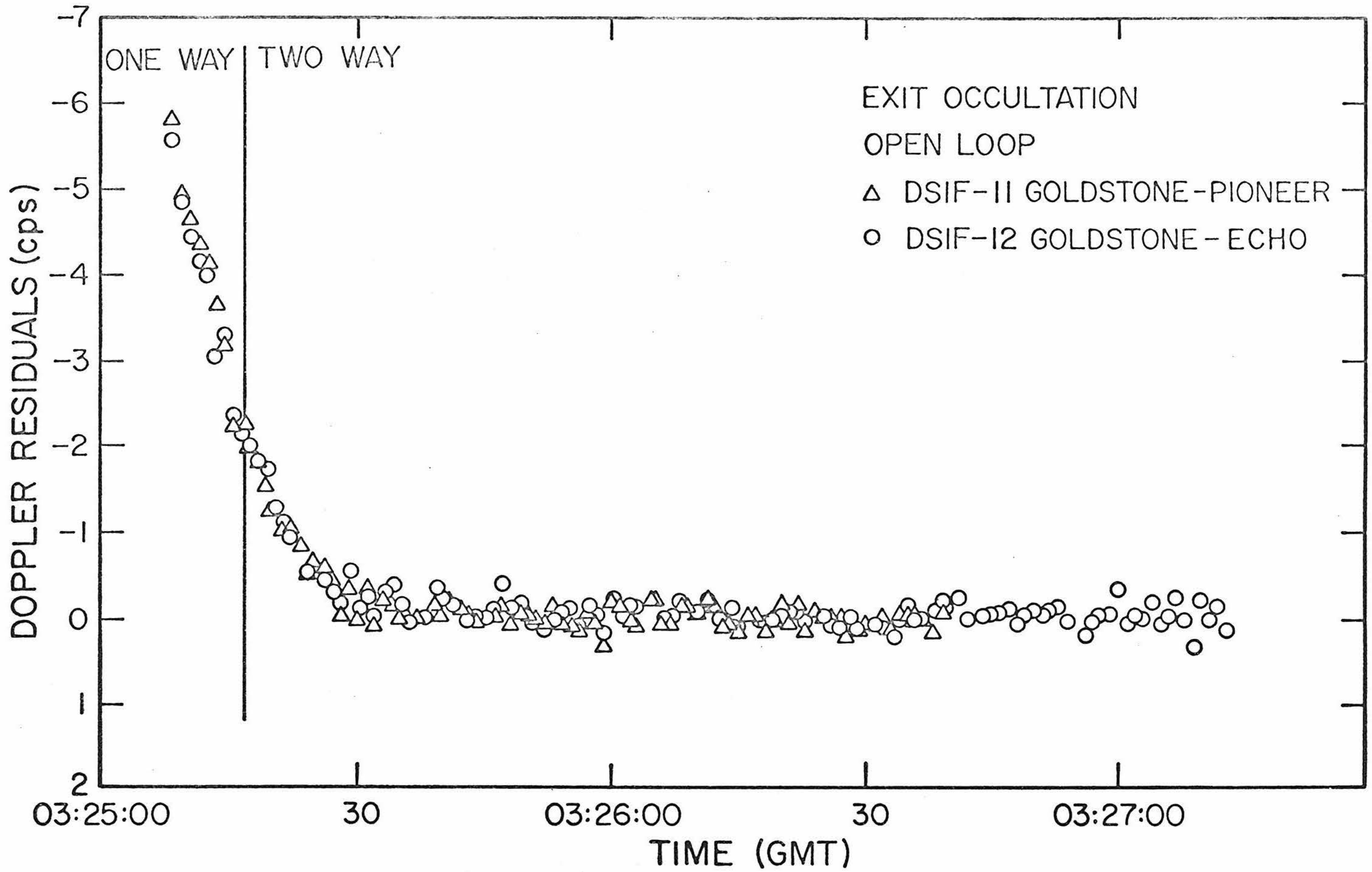


Fig. 5. Doppler Shift (Exit Residuals)

The measured deviation in phase of the telemetry signal was used to estimate the spatial characteristics of the index of refraction of the Martian atmosphere by a process of model fitting and integral inversion of the equation

$$\delta r(t) = \int_{\Gamma(t)} ds\, n(\underline{r}) - r_o(t) \qquad \text{(II.A.1)}$$

where

$\Gamma(t)$ = actual path followed by the telemetry signal

$n(\underline{r})$ = refractivity of the Martian atmosphere

$r_o(t)$ = straight-line path between the transmitting antenna on the space probe and the receiving antenna on the Earth.

Therefore, the deduced parameter from the occultation data was the refractivity of the Martian atmosphere as a function of altitude above the surface of the planet. From the refractivity of the atmosphere, one can further derive not only the scale height but also the pressure and density distributions of the atmosphere as a function of altitude above the surface of the planet. From the scale height of the atmosphere, the mean molecular weight can be determined and the composition of the atmosphere can be defined.

3. Model Atmospheres

The deduction of the properties of the Martian atmosphere from a knowledge of the refractivity of the atmosphere are discussed amply in the literature (3,4). The results of these model studies are presented in Table 1:

Property	Symbol	Dimension	VM-1	VM-2	VM-3	VM-4	VM-5	VM-6	VM-7	VM-8	VM-9	VM-10
Surface Pressure	p_o	mb	7.0	7.0	10.0	10.0	14.0	14.0	5.0	5.0	20.0	20.0
		lb/f^2	14.6	14.6	20.9	20.0	29.2	29.2	10.4	10.4	41.7	41.7
Surface Density	ρ_o	(gm/cc) 10^5	0.955	1.85	1.365	2.57	1.91	3.08	0.68	1.32	2.73	3.83
		(slugs/f^3) 10^5	1.85	3.59	2.65	4.98	3.7	5.97	1.32	2.56	5.30	7.44
Surface Temperature	T_o	°K	275	200	275	200	275	200	275	200	275	200
		°R	495	360	495	360	495	360	495	360	495	360
Stratospheric Temperature	T_s	°K	200	100	200	100	200	100	200	100	200	100
		°R	360	180	360	180	360	180	360	180	360	180
Acceleration of Gravity at Surface	g_s	cm/s^2	375	375	375	375	375	375	375	375	375	375
		f/s^2	12.3	12.3	12.3	12.3	12.3	12.3	12.3	12.3	12.3	12.3
Composition (percent)												
CO_2 (by mass)			28.2	100.0	28.2	70.0	28.2	35.7	28.2	100.0	28.2	13.0
CO_2 (by volume)			20.0	100.0	20.0	68.0	20.0	29.4	20.0	100.0	20.0	9.5
N_2 (by mass)			71.8	0.0	71.8	0.0	71.8	28.6	71.8	0.0	71.8	62.0
N_2 (by volume)			80.0	0.0	80.0	0.0	80.0	32.2	80.0	0.0	80.0	70.5
A (by mass)			0.0	0.0	0.0	30.0	0.0	35.7	0.0	0.0	0.0	25.0
A (by volume)			0.0	0.0	0.0	32.0	0.0	38.4	0.0	0.0	0.0	20.0
Molecular Weight	M	mol^{-1}	31.2	44.0	31.2	42.7	31.2	36.6	31.2	44.0	31.2	31.9
Specific Heat of Mixture	C_p	cal/gm °C	0.230	0.166	0.230	0.1530	0.23	0.174	0.230	0.166	0.230	0.207
Specific Heat Ratio	γ		1.38	1.37	1.38	1.43	1.38	1.45	1.38	1.37	1.38	1.41
Adiabatic Lapse Rate	Γ	°K/km	-3.88	-5.39	-3.88	-5.85	-3.88	-5.14	-3.88	-5.39	-3.88	-4.33
		°R/kf	-2.13	-2.96	-2.13	-3.21	-2.13	-2.82	-2.13	-2.96	-2.13	-2.38
Tropopause Altitude	h_t	km	19.3	18.6	19.3	17.1	19.3	19.4	19.3	18.6	19.3	23.1
		kf	63.3	61.0	63.3	56.1	63.3	63.6	63.3	61.0	63.3	75.8
Inverse Scale Height (stratosphere)	β	km^{-1}	0.0705	0.199	0.070	0.193	0.0705	0.1655	0.0705	0.199	0.0705	0.145
		$f^{-1} \times 10^5$	2.15	6.07	2.15	5.89	2.15	5.05	2.15	6.07	2.15	4.41
Free Stream Continuous Surface Wind Speed	$\bar{v}$	f/s	186.0	186.0	156.0	156.0	132.0	132.0	220.0	220.0	110.0	110.0
Maximum Surface Wind Speed	v_{max}	f/s	380.0	380.0	310.0	310.0	270.0	270.0	450.0	450.0	225.0	225.0
Design Gust Speed	v_g	f/s	200	200	150.0	150.0	150.0	150.0	200.0	200.0	100.0	100.0

Table 1. Model Atmospheres

from which it seems clear that the following characteristics of the Martian atmosphere are well established:

(1) Mars has a tenuous atmosphere. The molecular number density near the surface of the planet is only 0.7 to 1% that of the Earth.

(2) Carbon dioxide must be the principal atmospheric constituent in order to explain both the occultation and the spectroscopic measurements.

(3) The atmosphere is very cold at all altitudes. The temperature is about $180^{\circ} \pm 20^{\circ}K$ near the surface of the planet and is about $80^{\circ} \pm 20^{\circ}K$ at the height of the ionosphere.

(4) Because of the low temperature, the atmosphere is confined near the planet (the exosphere begins near 140 km) and the height profile of the atmospheric mass density is several orders of magnitude below that of the Earth at all altitudes even though the gravity is 62% lower on Mars.

In summary, the current models of the Martian atmosphere consist of a spherically layered atmosphere with a constant lapse rate lower layer and an isothermal (exponential) upper layer separated at the altitude of the tropopause. The adjustable parameters are the composition of the atmosphere; the surface pressure, density and temperature; the specific heat ratio; the constant lapse rate of the stratosphere; the exponential scale height of the isothermal layer; and the altitude of the tropopause.

Since there is still some uncertainty about the properties of the Martian atmosphere and the variation of these properties with altitude above the surface of Mars, two model atmospheres, VM-4 and VM-8, that have been chosen to represent a reasonable range of expected conditions, are used throughout the remainder of the study.

Since the various proposed model atmospheres assume a constant lapse rate lower layer in the stratosphere and an isothermal (exponential) upper layer, separated at the altitude of the tropopause, the pressure $p(h)$ of each atmospheric model is given by (5)

$$p = \begin{cases} p_o\left(1 + \frac{\Gamma}{T_o} h\right)^{\frac{\gamma}{\gamma-1}} & 0 \leq h \leq h_t \\ p_t\, e^{-\beta(h - h_t)} & h > h_t \end{cases} \tag{II.A.2}$$

and similarly, the density $\rho(h)$ of each model by (5)

$$\rho = \begin{cases} \rho_o\left(1 + \frac{\Gamma}{T_o} h\right)^{\frac{1}{\gamma-1}} & 0 \leq h \leq h_t \\ \rho_t\, e^{-\beta(h - h_t)} & h > h_t \end{cases} \tag{II.A.3}$$

where

p_o = surface pressure

ρ_o = surface density

T_o = surface temperature

γ = specific heat ratio

Γ = adiabatic lapse rate

h = altitude

β = inverse scale height

p_t = tropopause pressure

ρ_t = tropopause density

h_t = tropopause altitude

The appropriate values for these parameters are given in Table 1 for each model atmosphere.

Much of the information discussed in this part of the study has been graciously made available to the author by personal communication with the experimenters at the Jet Propulsion Laboratory.

B. The Entry-Trajectory

1. Introduction

The entry-trajectory of the capsule as it descends through the Martian atmosphere is developed in this section from a description of the entry capsule and the initial entry-trajectory data. To simplify the analysis that follows, the entry-trajectory is assumed to be linear, and is based on the two model atmospheres, VM-4 and VM-8, previously developed.

2. Capsule Description

The entry capsule (6) has a sphere-cone aerodynamic configuration, which ballistically decelerates the capsule within the Martian atmosphere. The spherical nose (nose radius = 0.1 diameter) is followed by a 60° half-angle cone forebody truncated in a knuckle section at the maximum diameter (corner radius = 0.05 diameter). The 0.1-diameter nose radius was chosen to minimize aerodynamic heating. The large cone half-angle of 60° was chosen to obtain the necessary deceleration in the tenuous Martian atmosphere. The corner radius of 0.05-diameter was based on tests conducted on other blunt body capsules and on some preliminary data available on 60° cones. This value reduces the heat problem at the corners.

The entry capsule is 6-1/2 f . in diameter and weighs 180 lbs. at entry. The ballistic coefficient* at entry is 0.12 slugs/ f^2 . The entry capsule uses a 400 MHz relay communication system that transmits at 8-W RF power during both the deflection maneuver and the entry phase. Also, an s-band direct-to-Earth communication system transmits at 3-W RF

---

*The ballistic coefficient is a measure of the distribution of mass per unit area and can be experimentally determined.

power at 2.295 GHz during the entry phase. The capsule is capable of carrying out an atmospheric entry experiment using pressure and temperature sensors, an accelerometer package, and a mass spectrometer. As the capsule descends through the Martian atmosphere it can perform direct measurements of the atmospheric properties and composition. Once on the surface of Mars, the lander can further obtain samples of the physical properties of the atmosphere at the surface of the planet, undertake soil composition experiments, and search for extraterrestrial life.

3. Initial Entry-Trajectory Data

The entry phase of a mission to the planet Mars begins nominally at an altitude of 800 kf above the surface of Mars just before the actual atmospheric entry, and ends with the impact on the surface of the planet. A set of initial conditions, consistent with the proposed space probe approach-trajectories, deflection manuever orientation, and deflection maneuver accuracies are used for the entry-trajectory analysis. These initial conditions are (6)·

(1) entry altitude: $h_e = 800$ kf

(2) entry velocity: $v_e = 22$ kf/s

(3) entry angle: $\psi_e = 55 \pm 6^{\circ}$

4. Linear Entry-Trajectory Characteristics

The predicted entry-trajectory of a blunt body capsule into the Martian atmosphere is depicted in Figure 6. Linear entry-trajectory theory is used to approximately determine the entry characteristics of the capsule. Those of interest in this study are the velocity and

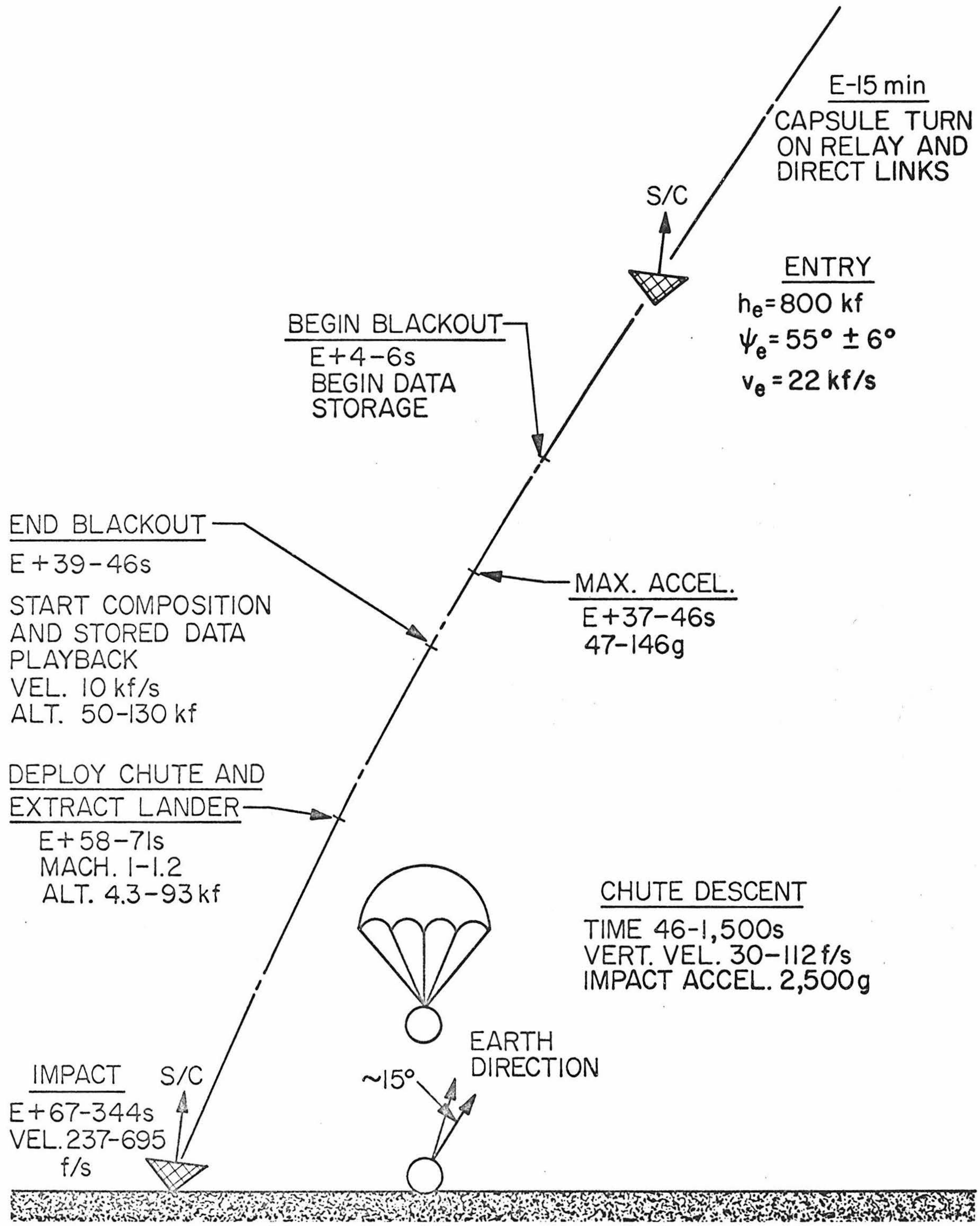


Fig. 6. Entry Profile

elapsed time-from-entry of the capsule, calculated as a function of altitude above the surface of Mars.

With the linear theory, the entry-trajectory of the capsule is assumed to be a straight-line path at the constant entry angle relative to the local horizontal at the assumed point of entry into the Martian atmosphere. The surface of Mars is assumed to be flat so that the altitude of the entry capsule above the surface of the planet is independent of the displacement of the capsule from the local vertical at the assumed point of entry into the atmosphere. Figure 7 is a schematic drawing of the linear entry trajectory showing the pertinent parameters and the resulting altitude error. Clearly, the accuracy of the linear approximation increases with increasing entry angle and is exact for an entry angle of 90°. At an entry angle of 30° or less, the curvature of Mars is no longer negligible in determining the velocity and elapsed time-from-entry profiles of the capsule. For a Martian entry, the predicted entry angle is 55°; therefore, the constant path angle assumption can be made without introducing any appreciable error into the analysis. This assumption will be advantageous in determining the entry profiles now to be discussed.

Since the path angle is assumed to be constant, the velocity equation (7) in the set of trajectory equations is readily integrated and yields

$$v = v_e \, e^{-\frac{p}{2g_s \Delta \sin \psi_e}} \tag{II.B.1}$$

The elapsed time-from-entry (7) is obtained by integrating the equation

$$\frac{dh}{dt} = - v \sin \psi \tag{II.B.2}$$

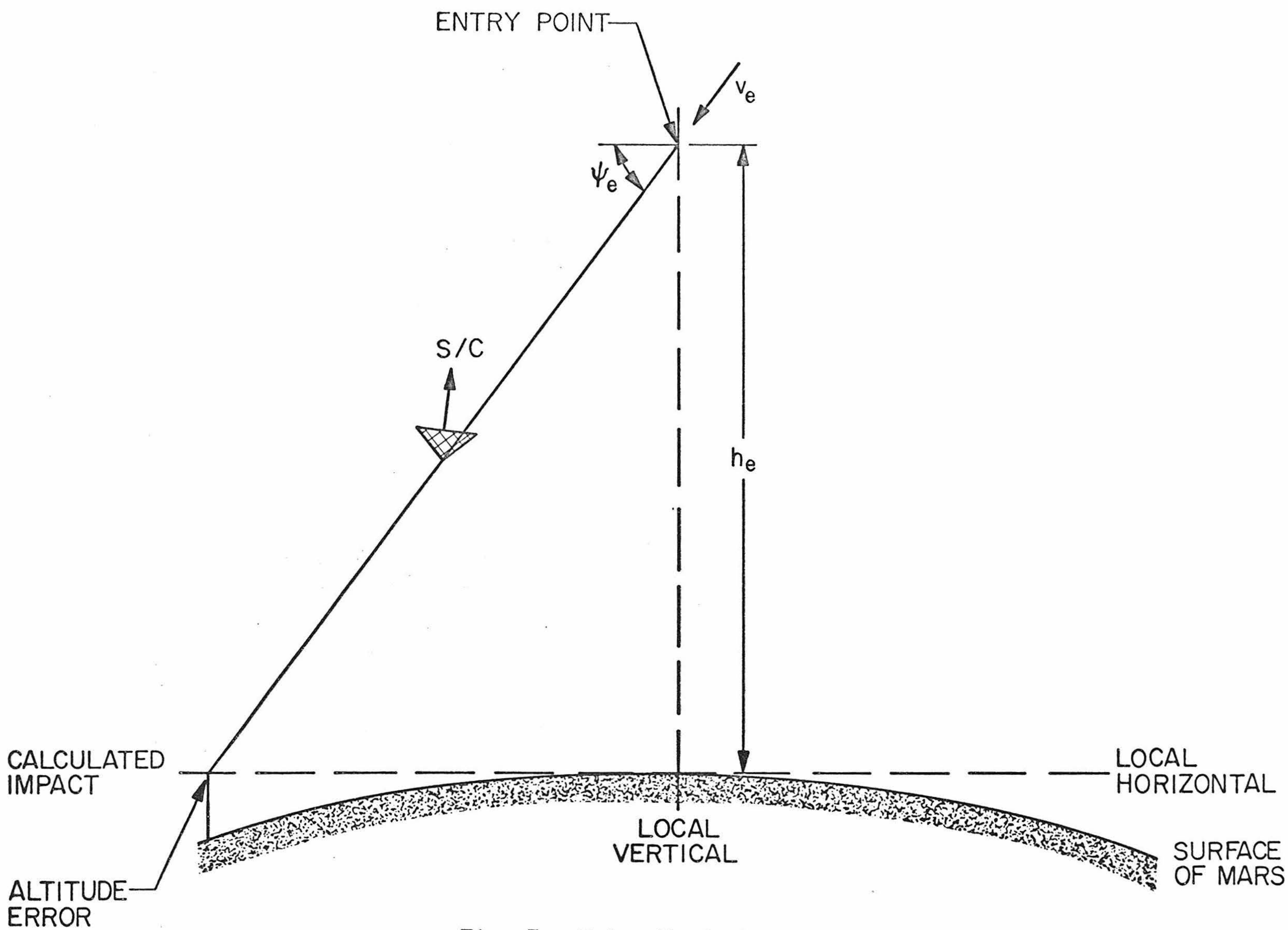


Fig. 7. Entry-Trajectory

with the aid of Eq. II.B.1 and the previous assumption that $\psi = \psi_e =$ constant. This results in the equation

$$\delta t = \frac{1}{\beta v_e \sin \psi_e} \left[Ei\left(\frac{p}{2g_s \Delta \sin \psi_e}\right) - Ei\left(\frac{p_e}{2g_s \Delta \sin \psi_e}\right) \right] \qquad \text{(II.B.3)}$$

where

v_e = entry velocity

ψ_e = entry angle

p_e = atmospheric pressure at the entry altitude

β = inverse scale height

g_s = nominal surface gravity

Δ = ballistic coefficient of the entry capsule

$$Ei(\zeta) \equiv \int_{-\infty}^{\zeta} d\eta \, \frac{e^{\eta}}{\eta} \qquad \zeta > 0 \quad \text{(exponential integral)}$$

Figures 8 and 9 are graphs of capsule velocity vs altitude above the surface of Mars, and capsule elapsed time-from-entry vs altitude above the surface of Mars. The results are for the VM-4 and VM-8 model atmospheres as obtained from Eqs. II.B.1 and II.B.3 for an entry altitude of 800 kf, an entry velocity of 22 kf/s, an entry angle of 55^{o}, and a ballistic coefficient of 0.12 slug/f^2.

An examination of Figure 9 reveals that for the VM-4 model atmosphere the capsule achieves terminal velocity at approximately 50s after entry. When this condition exists, Eq. II.B.3 is no longer valid and results in the unreasonably high value of 755.0s for the calculated impact time. Fortunately, only that part of the graph which lies below 50s is important to the work that follows.

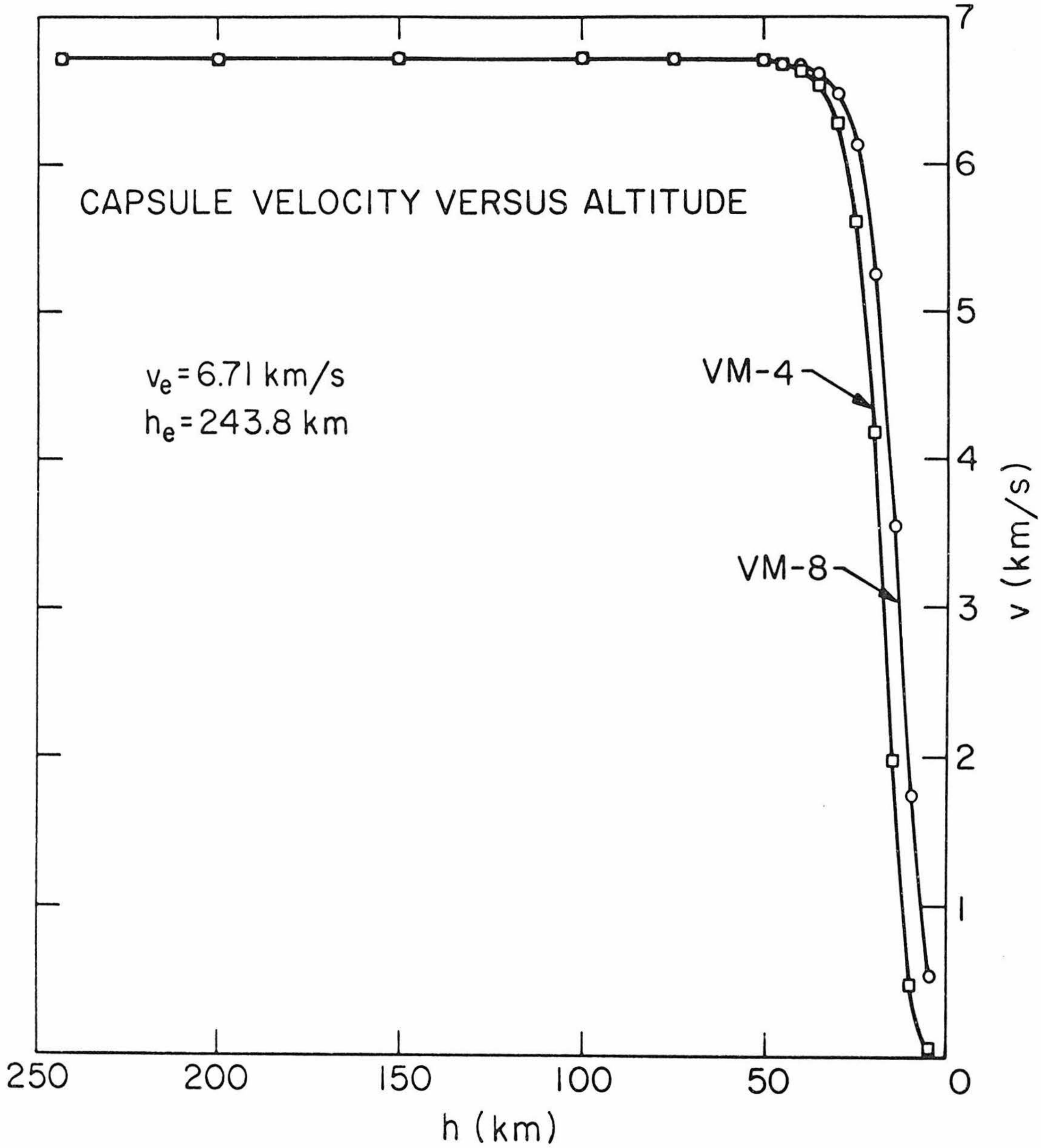


Fig. 8. Capsule Velocity vs. Altitude

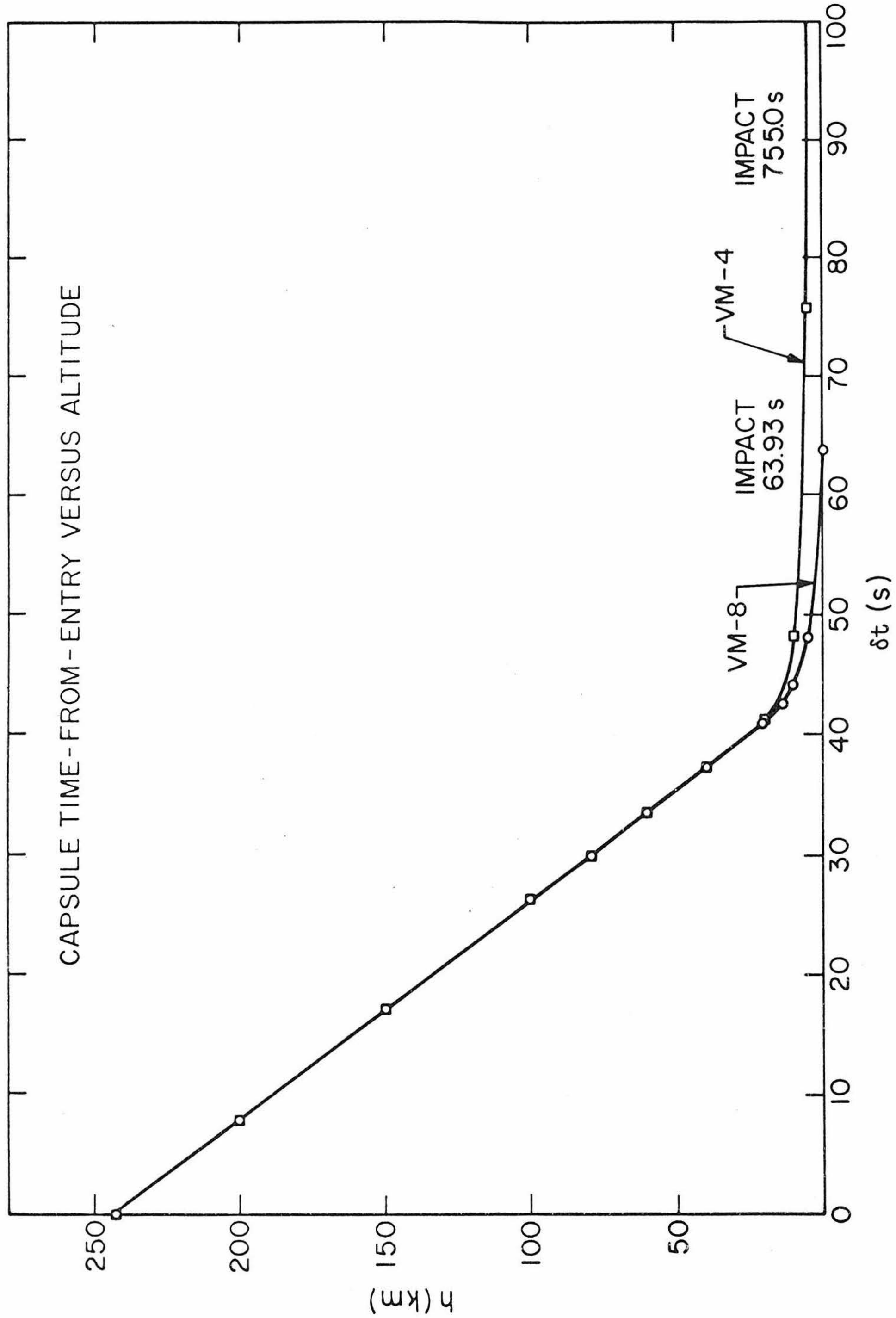


Figure 9. Capsule Time-from-Entry vs. Altitude

C. Plasma Profiles

1. Introduction

The radial dependence of the properties of the plasma in the wake region of the entry capsule is developed in this section from the assumed models of the Martian atmosphere and the entry-trajectory of the capsule as developed in the previous sections. These radial profiles are developed as a function of altitude above the surface of Mars and as a function of elapsed time-from-entry of the capsule.

The properties of interest in this study are the electron concentration and the collision frequency of the plasma. Actually, for a Mars entry, it has been shown that the collision frequencies are over three orders of magnitude below the signal frequencies of interest in this study and, therefore, can be neglected (8). Also, it can be determined from the entry-trajectory of the capsule in relation to the Earth at the time of entry that the propagation path will occur through the near wake only; therefore, only the electron concentration in the near wake is determined.

The radial profiles of the plasma in the near wake are developed first by examining the properties of the atmospheric gases in the stagnation region of the capsule, and then by analyzing the expansion of these gases as they flow out of the stagnation region, past the forebody of the capsule, and into the wake region. This analysis will determine only the peak electron concentration in the near wake as a function of altitude above the surface of Mars. Finally, the actual radial profiles of the plasma in the near wake are developed from experimental data furnished by the Jet Propulsion Laboratory.

2. Chemical Equilibrium-Normal Shock

Although many processes of interest in aerothermodynamics involve flow conditions in which high temperature gas mixtures are characterized by significant temperature gradients, and thus by non-equilibrium chemical phenomena, some processes are of interest for which either the temperature gradients disappear or the reaction rates are sufficiently fast to satisfy the assumption of chemical equilibrium.

If the principle of conservation of energy and the second law of thermodynamics are applied to a constant mass system in mechanical and thermal equilibrium at a particular pressure and temperature, the chemical composition of the gas mixture can be determined. Also if the assumption is made that the long range interaction between the particles of the gas are negligible, the individual constituents of the gas mixture can be described approximately by the ideal gas law. Then with this assumption the equations of mass action and mass balance for each constituent gas particle can be derived. A derivation of these equations is presented in the references (9). For completeness, the results are:

(1) mass action

$$\frac{n_j}{\prod_{k=s+1}^{s+m} n_k^{\alpha_{jk}}} = c_i (RT)^{-1+\sum_{k=s+1}^{s+m} \alpha_{jk}} \qquad (j = 1,2,\cdots,s) \qquad \text{(II.C.1)}$$

where

$$\ln c_j = -\frac{f_j^o}{RT} + \sum_{k=s+1}^{s+m} \alpha_{jk} \frac{f_k^o}{RT} \tag{II.C.2}$$

(2) mass balance

$$n_k + \sum_{j=1}^{s} n_j \alpha_{jk} = b_k \frac{\rho}{\rho_o} \quad (k = s+1, s+2, \cdots, s+m) \tag{II.C.3}$$

in which the quantities are defined as

m = number of chemical elements present

s = number of compounds present

b_i = concentration of the i^{th} element at STP

f_i^o = partial molal free energy of the i^{th} constituent at one atmosphere

n_i = concentration of the i^{th} constituent

R = universal gas constant (1.98726 cal/mole ^oK)

T = temperature

α_{jk} = number of atoms of the k^{th} element in the molecule of the j^{th} constituent

ρ = mass density of the mixture

ρ_o = mass density of the mixture at STP

After the chemical equilibrium concentrations of the gas mixture are determined, the thermodynamic properties of the gas mixture are calculated by the following relations

$$n = \sum_{i=1}^{s+m} n_i \tag{II.C.4}$$

$$\rho = \sum_{i=1}^{s+m} n_i m_i \tag{II.C.5}$$

$$h = \frac{\sum_{i=1}^{s+m} n_i \, h_i^o}{\rho} \qquad \text{(II.C.6)}$$

where

n = total concentration

ρ = total mass density

h = total enthalpy

h_i^o = partial molal enthalpy of the i^{th} constituent at one atmosphere

m_i = molecular weight of the i^{th} constituent

One important application of chemical equilibrium is the normal shock problem. The equations (10) governing the variation of the properties of the fluid through a normal shock, when the coordinate system is chosen so that the flow is steady and one-dimensional, and when the effects of body forces, diffusion, and radiative transfer are negligible, are

(1) continuity

$$\frac{d}{dz}(\rho v) = 0 \qquad \text{(II.C.7)}$$

(2) linear momentum

$$\rho v \frac{dv}{dz} + \frac{dp}{dz} = \frac{4}{3}\frac{d}{dz}\left(\eta \frac{dv}{dz}\right) \qquad \text{(II.C.8)}$$

(3) energy

$$\rho v \frac{dE}{dz} + p \frac{dv}{dz} = \frac{d}{dz}\left(K \frac{dT}{dz}\right) + \frac{4}{3}\eta\left(\frac{dv}{dz}\right)^2 \qquad \text{(II.C.9)}$$

where

p = pressure

ρ = mass density

v = velocity

$\dot{E}$ = energy

η = viscosity

K = thermal conductivity

If the fluid is assumed to be inviscid ($\eta \equiv 0$) and nonheat-conducting ($K \equiv 0$), or if the fluid is assumed to be in thermal equilibrium ($\frac{d}{dz} \equiv 0$), then the equations reduce to the form

$$\frac{d}{dz}(\rho v) = 0 \tag{II.C.10}$$

$$\rho v \frac{dv}{dz} + \frac{dp}{dz} = 0 \tag{II.C.11}$$

$$\rho v \frac{dE}{dz} + p \frac{dv}{dz} = 0 \tag{II.C.12}$$

Integrating these equations with respect to z yields

$$\rho v = k_1 \tag{II.C.13}$$

$$\rho v^2 + p = k_2 \tag{II.C.14}$$

$$\rho v(h + \frac{v^2}{2}) = k_3 \tag{II.C.15}$$

where k_1, k_2, and k_3 are the integration constants and h is the enthalpy per unit mass. If these equations are applied across a normal shock, then the equations become

$$\rho_1 v_1 = \rho_2 v_2 \tag{II.C.16}$$

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{II.C.17}$$

$$h_1 + \frac{v_1^2}{2} = h_2 + \frac{v_2^2}{2} \tag{II.C.18}$$

Equations II.C.16-18 relate the thermodynamic properties and the velocities in the upstream and downstream equilibrium regions of the flow associated with a moving normal shock. Mathematically, the problem is to satisfy the Hugoniot equation

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{II.C.19}$$

which is obtained from a suitable manipulation of Eqs. II.C.16-18. If the initial properties h_1, p_1, and ρ_1 are known, the chemical equilibrium computation yields relations for $h_2(\rho_2,T_2)$ and $p_2\ (\rho_2,T_2)$. Therefore, by selecting one property in region 2, the other property is determined by Eq. II.C.19.

3. Peak Electron Concentration (Stagnation Region)

In the dense lower altitudes of the Martian atmosphere, where the chemical reaction rates in the flow regime of an entry capsule are sufficiently fast, the free-flight entry calculation is one of supplying thermodynamic properties and chemical compositions of a multicomponent, high temperature, real gas mixture, which approximately satisfies the conditions of chemical equilibrium. In the less dense, higher altitudes of the Martian atmosphere, the assumption of chemical equilibrium is no longer justified. However, an equilibrium solution at the less dense altitudes would produce a conservative estimate of the properties of the gas mixture and, therefore, will be in keeping with the objectives of this study.

Also, to a close approximation, the properties of the gas mixture in the stagnation region of the capsule are similar to that behind a traveling normal shock.

The Hugoniot equation and auxiliary equations derived from Eqs. II.C.16 and II.C.17 with subscripts appropriate to the free-flight moving shock problem are

$$h_s - h_\infty = \frac{1}{2}(p_s - p_\infty)\left(\frac{1}{\rho_\infty} + \frac{1}{\rho_s}\right) \tag{II.C.20}$$

$$v_c = \sqrt{\frac{\rho_s}{\rho_\infty}\frac{p_s - p_\infty}{\rho_s - \rho_\infty}} \tag{II.C.21}$$

$$v_{g/c} = \sqrt{\frac{\rho_\infty}{\rho_s}\frac{p_s - p_\infty}{\rho_s - \rho_\infty}} \tag{II.C.22}$$

$$h_{sr} = h_s + \frac{1}{2}v_{g/c}^2 \tag{II.C.23}$$

where

$(\ )_s$ = shock region

$(\ )_\infty$ = free stream region

The terms v_c and $v_{g/c}$ denote the velocity of the capsule and the velocity of the gases relative to the capsule, respectively, and h_{sr} is the enthalpy of the stagnation region.

To obtain the properties of the s+m constituents of a gas mixture in chemical equilibrium, the m mass balance equations and the s nonlinear mass action equations must be solved simultaneously. A method has been developed to solve these equations numerically. It expands the mass action and mass balance equations in a Taylor's series about a point defined by a set of concentrations which is assumed to approximately satisfy the mass action and mass balance conditions. If terms higher than the first order are neglected, a system of linear equations is obtained that approximates the mass action and mass balance equations. The solution of the linear system of equations results in a new set of concentrations, which are closer to the actual solution to the mass action and mass balance equations. If this procedure is repeated, a solution with any desired accuracy can be obtained.

The solution to the free-flight moving shock problem consists of finding a thermodynamic state defined by ρ_s and T_s that satisfies the Hugoniot equation. The initial conditions to the Hugoniot equation are specified by the pressure, temperature, and concentrations as found by the method above. To find the final state that satisfies the Hugoniot equation, a final temperature T_s is chosen, and the value of ρ_s is varied until values of p_s and h_s are found which satisfy the Hugoniot equation. After the properties which satisfy the Hugoniot equation are found, the velocity of the capsule and the velocity of the gases relative to the capsule are found from Eqs. II.C.22 and II.C.23. Then the properties of the gases in the stagnation region of the capsule are determined.

An equilibrium thermochemistry and normal shock computer program (11) has been developed to perform the above computations. The input parameters to the program are

initial mixture constituents

initial mixture compositions

initial molal enthalpies

initial free stream pressure

initial free stream temperature

shock temperature

The output of the program is a complete chemical and thermodynamic description of the gases behind the moving normal shock. The output parameters of interest for this study are the shock velocity and electron concentration in the stagnation region of the capsule.

It is convenient to use the altitude above the surface of Mars as an input parameter in the following computations performed with the equilibrium thermochemistry and normal shock computer program. Since the program input is in terms of the initial free stream pressure instead of the altitude, Eq. II.A.2 is used to convert each altitude of interest into an equivalent free stream pressure.

Because the input value of the shock temperature cannot be determined in advance, the electron concentration in the stagnation region of the capsule is calculated for a series of shock temperatures. The program output then gives the appropriate electron concentration and shock velocity corresponding to each shock temperature in the series, and a curve of electron concentration vs. shock velocity is developed for each initial free stream pressure, with the shock temperature as a parameter along the curve. The correct shock velocity for each curve is the velocity of the capsule for that curve, and the capsule velocity corresponding to the free stream pressure along each curve is given by Eq. II.B.1 . Therefore, with the use of Eq. II.B.1, the electron concentration in the stagnation region of the capsule can be determined as a function of capsule velocity by linear interpolation, when necessary. The construction of one such curve with all the pertinent parameters is shown in Figure 10.

Figure 11 contains graphs of the electron concentration in the stagnation region of the capsule vs. shock velocity or capsule velocity. The graphs were obtained for the VM-4 and VM-8 model atmospheres from the thermochemistry and normal shock computer program by the method just described. The curves used to construct these graphs are not shown.

4. Peak Electron Concentrations (Wake Region)

The peak electron concentration in the wake region of the entry capsule is now determined since the antenna is mounted on the aft part of the capsule and the propagation path is through the near wake.

An exact analysis of the expansion of the gases flowing out of the stagnation region, past the forebody of the capsule, and into the wake region is not attempted here because the solution would require the use of nonequilibrium chemistry. Instead, a frozen flow approximation to the actual flow conditions is used to estimate the values of the peak electron concentrations in the wake region. Other chemical equilibrium schemes are possible, but because the frozen flow approximation yields

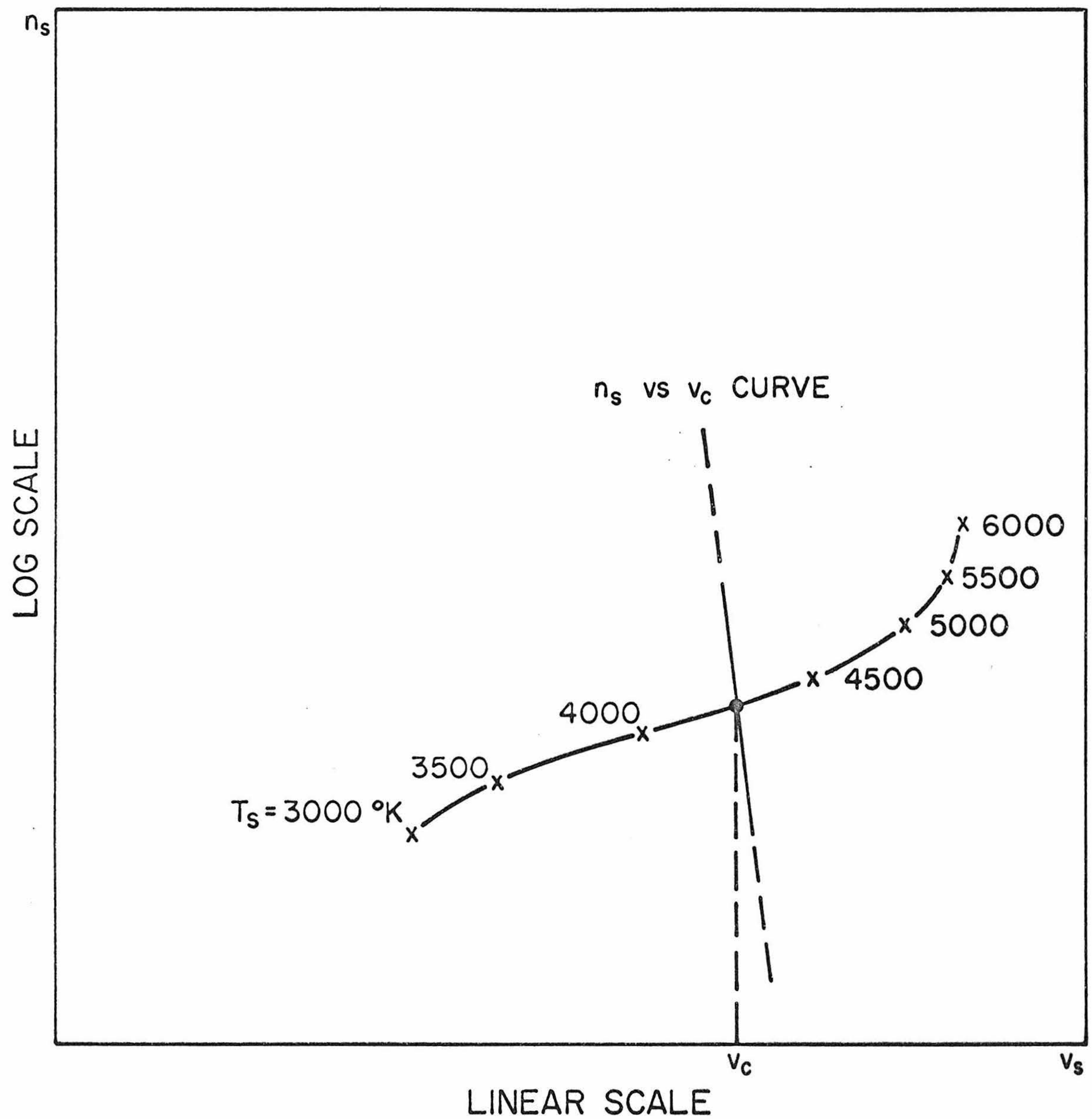


Fig. 10. Shock Temperature Curve

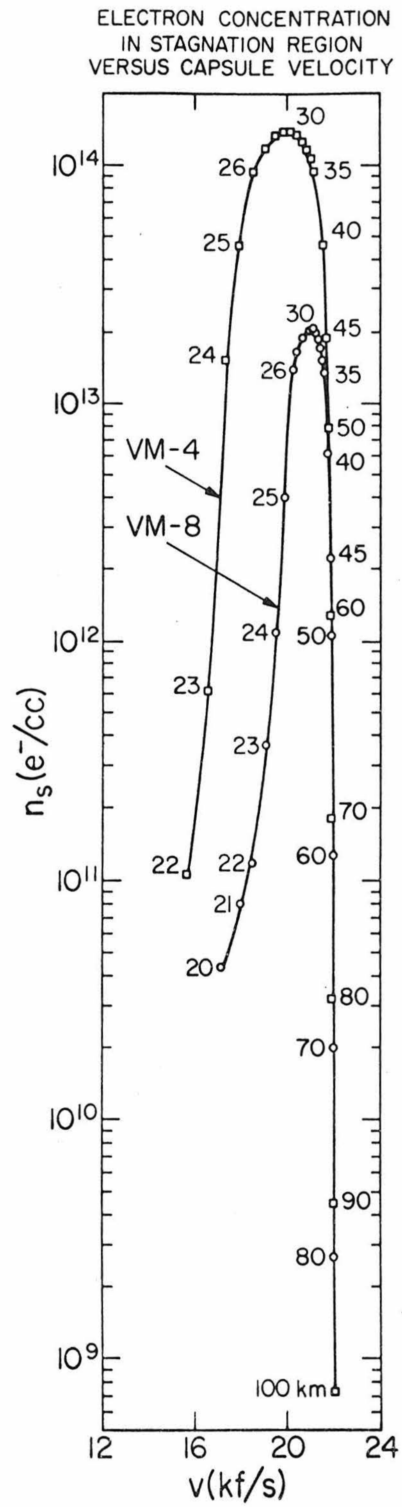


Fig. 11. Electron Concentration in Stagnation Region vs. Capsule Velocity

the greatest value for the electron concentration in the wake region, it is in keeping with the limitations of this study.

In the frozen flow approximation it is assumed that the total number of electrons formed in the stagnation region remains constant as the gases flow around the capsule and into the wake region. Any change in the electron concentration is due to the expansion of the gases as they flow into the wake region. Although the value of the specific heat ratio is not a constant during the expansion of the gases, the results are not sensitive to the value selected (8). In the frozen flow approximation the value of the electron concentration n_w in the wake region is related to the value of the electron concentration n_s in the stagnation region by (8)

$$n_w = n_s \left(\frac{p_\infty}{p_s}\right)^{1/\gamma} \tag{II.C.24}$$

where the pressure ratio (p_s/p_∞) across the shock is obtained from the output of the equilibrium thermochemistry and normal shock computer program. The term γ denotes the specific heat ratio.

Graphs of the peak electron concentration in the wake region vs. altitude above the surface of Mars, and graphs of the peak electron concentration in the wake region vs. elapsed time-from-entry of the capsule are obtained from the results contained in Figure 11 by using Eq. II.C.24 and relationships between altitude, capsule velocity, and capsule elapsed time-from-entry as given by Eq. II.B.1 and II.B.3 . These graphs are shown in Figures 12 and 13 for the VM-4 and VM-8 model atmospheres.

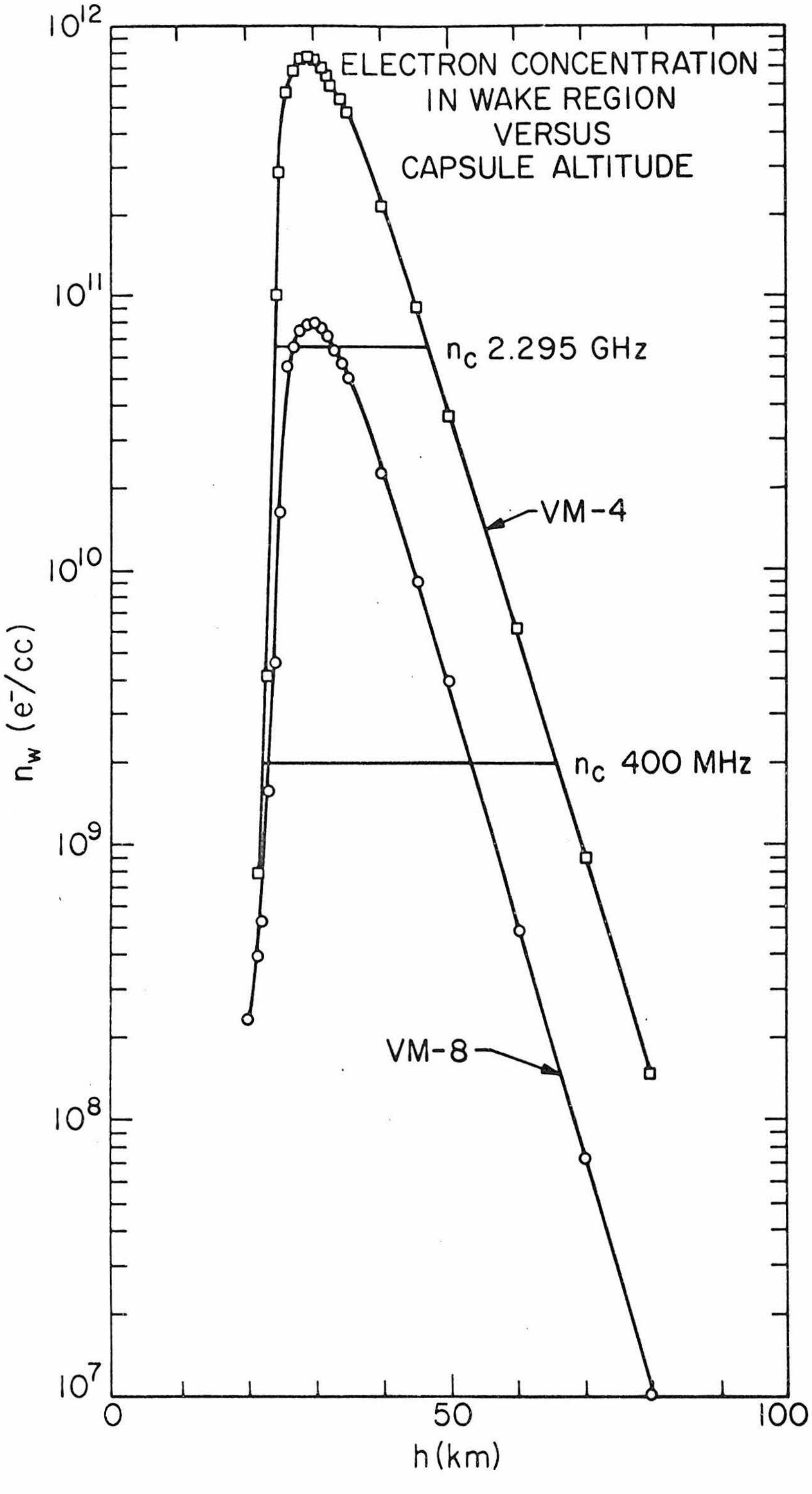


Fig. 12. Electron Concentration in Wake Region vs. Capsule Altitude

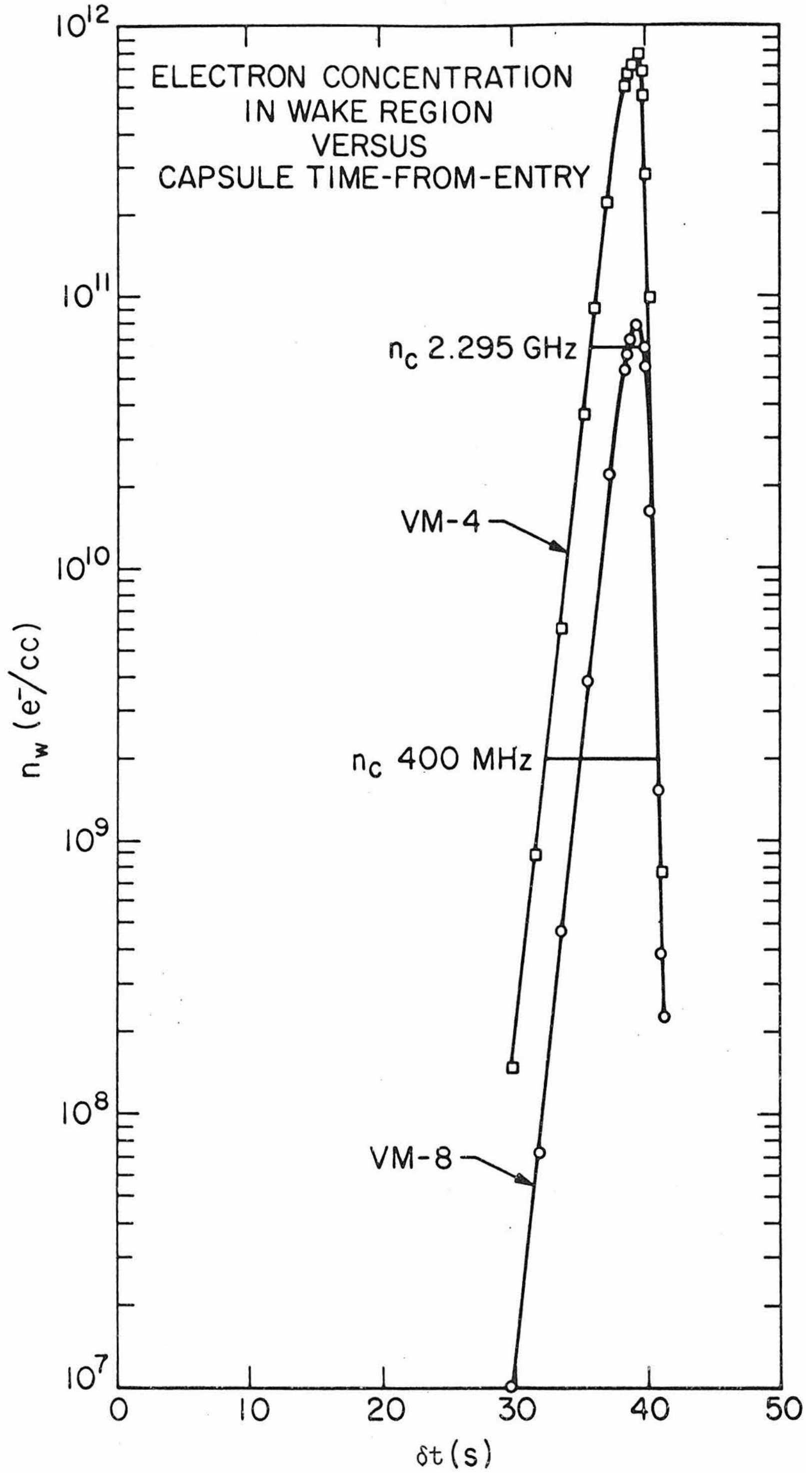


Fig. 13. Electron Concentration in Wake Region vs. Capsule Time-from-Entry

For the nonconducting, isotropic plasma present on a Mars entry, a critical value of the electron concentration n_c can be determined such that, if $n_w < n_c$, the plasma is underdense and waves in the plasma propagate without attenuation; and, if $n_w > n_c$, the plasma is overdense and waves in the plasma are evanescent and carry no power. Therefore, blackout will occur during a Mars entry whenever $n_w > n_c$.

The critical electron concentration for 400 MHz is 1.99×10^9 e^-/cc and the critical electron concentration for 2.295 GHz is 6.53×10^{10} e^-/cc. These are shown in Figs. 12 and 13. Clearly, blackout occurs during a Mars entry, and the altitude at which blackout begins and ends and the duration of blackout can be determined from Figures 12 and 13. A discussion of these aspects of blackout is presented in the references (12).

In the remainder of this study, the effects of the ionized wake on wave propagation before and after blackout will be investigated.

5. Radial Distributions

Typical electron concentration profiles in the wake region of an entry capsule are shown in Figure 14. These profiles were obtained experimentally by the Jet Propulsion Laboratory from shock tube tests performed with blunt body capsules. In the figure the electron concentrations are plotted against the distance off the dividing streamline* perpendicular to the axis of symmetry of the capsule, for various distances downstream from the aft part of the capsule.

The electron concentration profiles are determined as a function of the peak value of the electron concentration in the wake region of the capsule. It has been found from these experimental tests that the shapes of the distributions are not sensitive to variations in the peak value of the electron concentration.

Since the peak value of the electron concentration in the wake region of the capsule has been determined in the previous sections of this study, the radial profiles of the electron concentration in the near wake of the capsule can be determined from Figure 14. The profile labeled $z = 0.5$ in Figure 14 is typical of the electron concentrations found in the near wake of the capsule and will be used as a model to represent the actual near wake present on a Mars entry.

---

*The dividing streamline is the path taken by the gases closest to the capsule at the maximum diameter as they flow into the wake region.

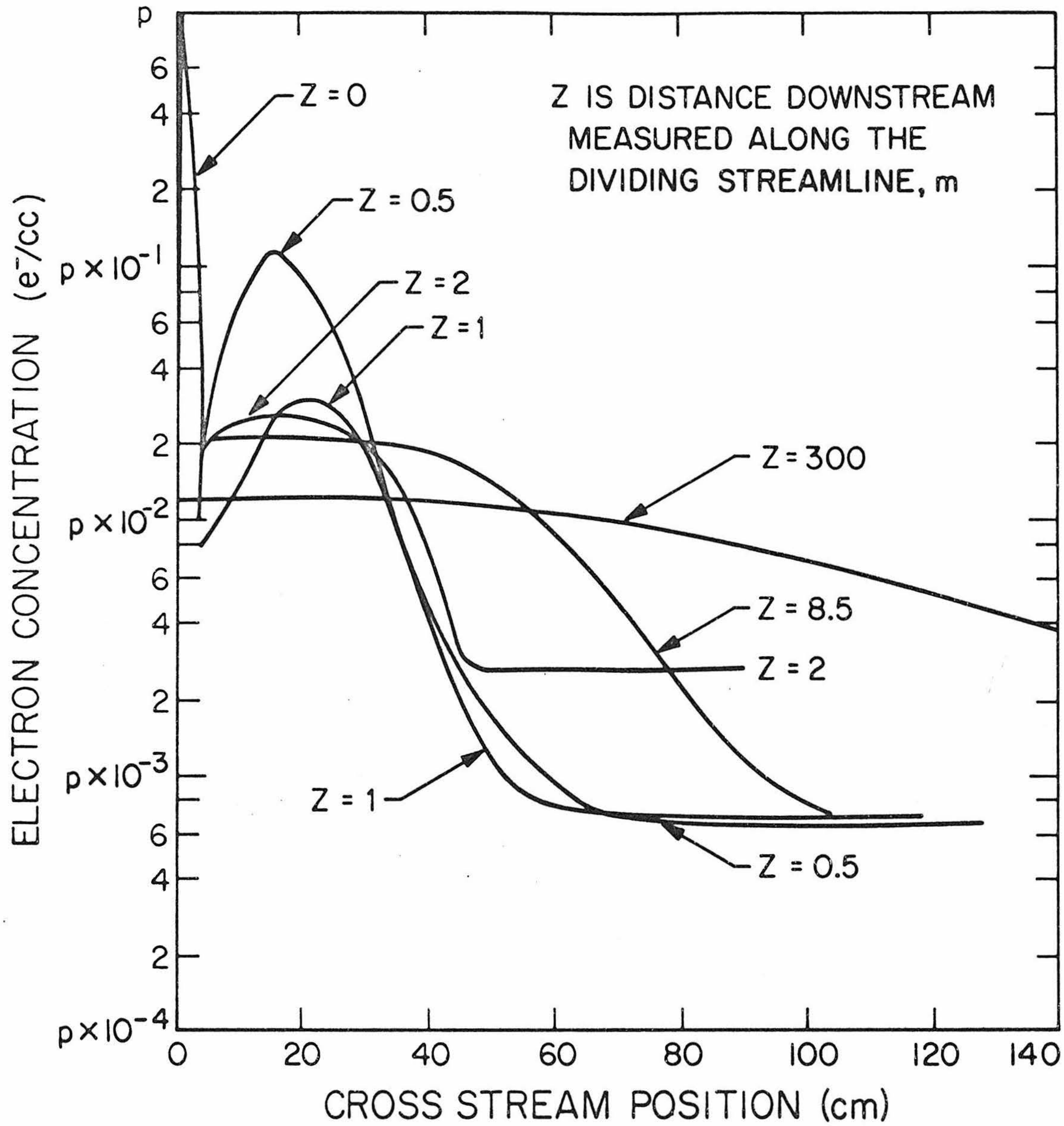


Fig. 14. Typical Electron Concentration Profiles in Wake Region

III. THE ELECTROMAGNETIC PROBLEM

A. Problem Statement

The effects of a moving plasma on electromagnetic wave propagation are investigated in this section. The plasma in its most general state is assumed to be inhomogeneous, ansiotropic*, and conducting.

As noted in the previous sections, no magnetic field is detectable on Mars, and the collision frequencies of the plasma are over three orders of magnitude below the signal frequencies of interest in this study. Therefore, for the Martian atmosphere, the plasma can be assumed to be isotropic and nonconducting.

To make this study applicable to other extraterrestrial planets, which may or may not have the same properties as Mars, the following work will be undertaken considering the more general case in which the anisotropy and conductivity of the plasma are present.

The electron concentration profile in the wake region of the capsule possesses axial symmetry about the center line of the capsule. As shown in Figure 14, the electron concentration is also a slowly varying and continuous function of the radial and axial distances away from the capsule. For increasing axial distance behind the capsule, there is a decay in the peak electron concentration and an expansion of the wake radius. Since these axial changes are much slower than the radial changes, they are not expected to alter significantly the results obtained from assuming that the inhomogeneity is a function of

---

*The origin of the magnetic bias that gives rise to the anisotropy of the plasma is considered to be outside the scope of this study.

the radius only. Therefore, the wake region of the capsule is approximated by a cylindrically stratified plasma shell consisting of n homogeneous plasma layers. The i^{th} layer of the plasma $(1 \leq i \leq n)$, which is described by the electron concentration n_i' and the collision frequency f'_{ci}, is biased on the z direction by an applied magnetic field $\underline{b}_{oi}$ and is moving in the z direction with a velocity $\underline{v}_i$. The geometry of the problem is shown in Figure 15.

The antenna in this study is mounted $\lambda_o/4$ * above the aft part of the capsule and emits right circularly polarized radiation in the z direction. This antenna is represented by a turnstile antenna located $\lambda_o/4$ above an infinite ground plane. Although the aft part of the capsule is finite, it is large compared with the wavelengths of interest in this study; consequently, the diffraction effects of the finite capsule can be neglected and the assumption of an infinite ground plane is reasonable.

Since the radiation from a turnstile antenna with an infinite ground plane can be constructed from a knowledge of the radiation from a horizontal dipole, the problem reduces to one of finding the fields of a horizontal dipole located as shown in Figure 15.

To analyze the effects of the moving plasma on the radiation from the antenna, the theories of Minkowski's phenomenological electrodynamics of a moving medium are used to derive the required field equations in the moving plasma. This approach is based on the covariance of Maxwell's equations and on the invariance of the constitutive parameters of the plasma when Lorentz-transforming the field equations between inertial reference frames.

---

* λ_o is the free space wavelength of the radiation emitted by the antenna.

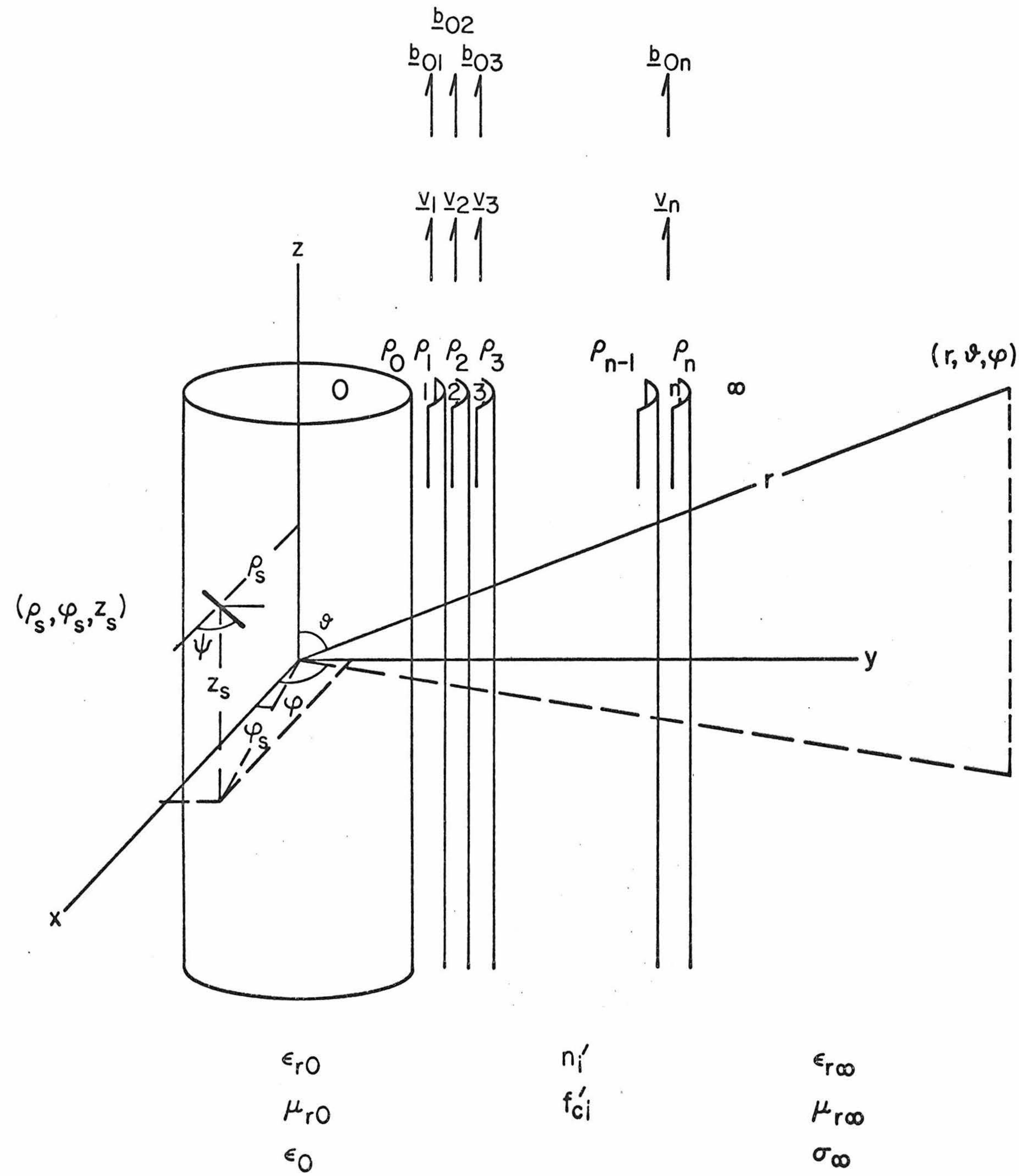


Fig. 15. Near Wake Geometry

One frame of reference is chosen to be at rest with respect to the plasma in each layer, and another frame of reference is chosen to be at rest with respect to the antenna. From the point of view of the rest frame of the plasma in each layer, the problem is one of solving the inhomogeneous wave equation in a stationary, anisotropic, and conducting plasma. After the stationary wave equation in each layer of the plasma has been solved, the resulting integral expressions for the cylindrical components of the field vectors are Lorentz-transformed into the rest frame of the antenna. Then, in the rest frame of the antenna, the boundary conditions on the tangential components of the field vectors are satisfied. Finally, the complete integral expressions for the spherical components of the field vectors are evaluated using the techniques of asymptotic expansions to yield the radiation patterns of the antenna.

B. Stationary Media

1. Field Equations

In the following discussion let any vector $\underline{v}$ represent a function of the time t and of the space $\underline{r}$, i.e., $\underline{v} \equiv \underline{v}(t,\underline{r})$. Then Maxwell's equations in a stationary medium

$$\underline{\nabla} \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{III.B.1}$$

$$\frac{1}{\mu_o} \underline{\nabla} \wedge \underline{B} = \underline{j}_t + \varepsilon_o \frac{\partial \underline{E}}{\partial t} \tag{III.B.2}$$

$$\varepsilon_o \underline{\nabla} \cdot \underline{E} = \rho_t \tag{III.B.3}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{III.B.4}$$

describe the electromagnetic field by the vectors $\underline{E}$ and $\underline{B}$ and characterize the medium by the total charge density ρ_t and the total current density $\underline{j}_t$. The terms μ_o and ε_o denote, respectively, the permeability and permittivity of the vacuum.

The total charge and current density terms can be separated into two distinct parts (13)

$$\rho_t = \rho_o + \rho_i \tag{III.B.5}$$

$$\underline{j}_t = \underline{j}_o + \underline{j}_i \tag{III.B.6}$$

where

$$\rho_o \equiv \rho_a \quad \text{(applied)} \tag{III.B.7}$$

$$\rho_i \equiv -\underline{\nabla} \cdot \underline{P} + \frac{1}{2} \underline{\nabla}\, \underline{\nabla} : \underline{\underline{Q}} + \cdots \quad \text{(induced)} \qquad \text{(III.B.8)}$$

$$\underline{j}_o \equiv \begin{cases} \underline{j}_a & \text{(applied)} \\ \underline{j}_{cond} & \text{(conduction)} \\ \underline{j}_{conv} & \text{(convection)} \end{cases} \qquad \text{(III.B.9)}$$

$$\underline{j}_i \equiv \frac{\partial \underline{P}}{\partial t} - \frac{1}{2} \frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{\underline{Q}} + \underline{\nabla} \wedge \underline{M} + \cdots \text{(induced)} \qquad \text{(III.B.10)}$$

The terms $\underline{P}$, $\underline{\underline{Q}}$, $\underline{M}$, etc. denote the volume densities of the induced multipole moments that are produced by the effect of the electromagnetic field on the neutral particles of the medium and are, therefore, functions of the field vectors $\underline{E}$ and $\underline{B}$.

The electric field vector $\underline{D}$ is defined by

$$\underline{\nabla} \cdot \underline{D} = \rho_o \qquad \text{(III.B.11)}$$

or, after comparing this relationship with Eqs. III.B.3 and III.B.5, by

$$\underline{D} \equiv \varepsilon_o \underline{E} + \underline{P} - \frac{1}{2} \underline{\nabla} \cdot \underline{\underline{Q}} + \cdots \qquad \text{(III.B.12)}$$

Similarly, the magnetic field vector $\underline{H}$ is defined by

$$\underline{\nabla} \wedge \underline{H} = \underline{j}_o + \frac{\partial \underline{D}}{\partial t} \qquad \text{(III.B.13)}$$

or, after comparing this relationship with Eqs. III.B.2 and III.B.6, by

$$\underline{H} \equiv \frac{\underline{B}}{\mu_o} - \underline{M} + \cdots \qquad \text{(III.B.14)}$$

It is convenient to express the field vectors in the following form:

$$\underline{\upsilon}(t,\underline{r}) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \underline{v}(\omega,\underline{r})\ e^{-i\omega t} \qquad \text{(III.B.15)}$$

$$\underline{v}(\omega,\underline{r}) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \underline{\upsilon}(t,\underline{r})\ e^{i\omega t} \qquad \text{(III.B.16)}$$

which is just a Fourier integral transformation in t and ω .

In what follows, let any transformed vector $\underline{v}$ represent a function of the transformed variable ω and of the space $\underline{r}$, i.e., $\underline{v} \equiv \underline{v}(\omega,\underline{r})$. Consequently, Maxwell's equations in terms of the transformed variables are

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o \underline{\underline{\eta}} \cdot \underline{H} \qquad \text{(III.B.17)}$$

$$\underline{\nabla} \wedge \underline{H} = \underline{j}_o - i\omega\varepsilon_o \underline{\underline{\zeta}} \cdot \underline{E} \qquad \text{(III.B.18)}$$

$$\varepsilon_o \underline{\nabla} \cdot \underline{\underline{\zeta}} \cdot \underline{E} = \rho_o \qquad \text{(III.B.19)}$$

$$\mu_o \underline{\nabla} \cdot \underline{\underline{\eta}} \cdot \underline{H} = 0 \qquad \text{(III.B.20)}$$

where it is assumed that a linear relationship exists between the multipole moments and the transformed field vectors such that equations III.B.12 and III.B.14 transform into the expressions

$$D \equiv \underline{\underline{\varepsilon}} \cdot \underline{E} \equiv \varepsilon_o \underline{\underline{\zeta}} \cdot \underline{E} \qquad \text{(III.B.21)}$$

$$\underline{B} \equiv \underline{\underline{\mu}} \cdot \underline{H} \equiv \mu_o \underline{\underline{\eta}} \cdot \underline{H} \qquad \text{(III.B.22)}$$

$\underline{\underline{\zeta}}$ is the relative permittivity dyad of the medium and $\underline{\underline{\eta}}$ is the relative permeability dyad of the medium.

2. Conduction-Convection Currents

The conduction current density in a stationary dielectric is given simply by the Ohm's law

$$\underline{j}_{cond} = \underline{\underline{\sigma}} \cdot \underline{E} \tag{III.B.23}$$

where†

$$\underline{\underline{\sigma}} = \underline{\underline{u}}\, \sigma \tag{III.B.24}$$

The term σ denotes the conductivity of the dielectric.

A suitable model of a stationary plasma, consistent with the objectives of this study, is that of a certain number n of electrons per unit volume free to move under the influence of an applied electromagnetic field and a static magnetic field, but subject to a damping force due to collisions characterized by a damping constant ω_c.*

Only the interaction between the wave and the free electrons need be considered for the frequencies of interest in this study. The convection current density in a stationary plasma is determined by examining the motion of the free electrons. From Newton's second law of motion and the Lorentz force equation, the equation of motion of the free electrons is

$$nm \frac{d\underline{v}}{dt} = nq(\underline{E} + \underline{v} \wedge \underline{B}) + nq\, \underline{v} \wedge \underline{B}_o - nm\, \omega_c \underline{v} \tag{III.B.25}$$

In the present case the nonlinear $\underline{v} \wedge \underline{B}$ term is dropped, since

---

*The damping constant ω_c represents the average number of collisions the electrons undergo per unit time.

† $\underline{\underline{u}}$ is the unit dyad.

$|\underline{v} \wedge \underline{B}| << |\underline{E}|$. Also, it is assumed that the static magnetic field applied to the plasma is in the z direction, i.e., $\underline{B}_o \equiv \hat{z} b_o$ and $b_o \neq 0$. Then in the transformed space, Eq. III.B.25 can be rewritten in the following equivalent but preferred ways:

$$(-i\omega + \omega_c)\underline{v} + \omega_g \hat{z} \wedge \underline{v} = \frac{q}{m}\underline{E}$$

$$(-i\omega + \omega_c)\underline{v} + \omega_g \underline{v}^s = \frac{q}{m}\underline{E} \qquad \text{(III.B.26)}$$

$$[(-i\omega + \omega_c)\underline{\underline{u}} + \omega_g \underline{\underline{c}}] \cdot \underline{v} = \frac{q}{m}\underline{E}$$

where ω_g is the gyrofrequency of the free electrons defined as

$$\omega_g \equiv \frac{q}{m} b_o \qquad \text{(III.B.27)}$$

The properties of the projection operator $(\;)^s$ and the dyad $\underline{\underline{c}}$ are described fully in the appendix. The projection operator $(\;)^s$ or the dyad $\underline{\underline{c}}$ essentially reduces the vector cross product to a scalar dot product.

Let the linear dyadic operator $\underline{\underline{O}}(\omega)$ be defined as

$$\underline{\underline{O}}(\omega) \equiv (-i\omega + \omega_c)\underline{\underline{u}} + \omega_g \underline{\underline{c}} \qquad \text{(III.B.28)}$$

---

*A rigorous derivation of Eq. III.B.26 using a statistical distribution to describe the plasma can be found in the Theory of Wave Propagation by C. H. Papas (13).

Then

$$\underline{v} = \frac{q}{m} \underline{\underline{\mathcal{O}}}^{-1}(\omega) \cdot \underline{E} \tag{III.B.29}$$

where $\underline{\underline{\mathcal{O}}}^{-1}(\omega)$ is the inverse of $\underline{\underline{\mathcal{O}}}(\omega)$, i.e.,

$$\underline{\underline{\mathcal{O}}}(\omega) \cdot \underline{\underline{\mathcal{O}}}^{-1}(\omega) = \underline{\underline{u}} \tag{III.B.30}$$

The inverse of $\underline{\underline{\mathcal{O}}}(\omega)$ can be determined by examining its individual components in matrix form. This process yields

$$\underline{\underline{\mathcal{O}}}^{-1}(\omega) = \frac{(-i\omega + \omega_c)\underline{\underline{u}} - \omega_g \underline{\underline{c}} + \dfrac{\omega_g^2}{-i\omega + \omega_c} \hat{z}\,\hat{z}}{(-i\omega + \omega_c)^2 + \omega_g^2} \tag{III.B.31}$$

Therefore, from Eqs. III.B.29 and III.B.31, the average velocity of the free electrons can be written as

$$\underline{v} = i \frac{q}{m} \frac{(\omega + i\omega_c)\underline{\underline{u}} - i\omega_g \underline{\underline{c}} - \dfrac{\omega_g^2}{\omega + i\omega_c} \hat{z}\,\hat{z}}{(\omega + i\omega_c)^2 - \omega_g^2} \cdot \underline{E} \tag{III.B.32}$$

The convection current density in the plasma is defined as

$$\underline{j}_{conv} = nq\,\underline{v} \tag{III.B.33}$$

and since $\underline{v}$ is related to $\underline{E}$ by Eq. III.B.32, then

$$\underline{j}_{conv} = i\omega\varepsilon_o \frac{\omega_p^2}{\omega} \frac{(\omega + i\omega_c)\underline{\underline{u}} - i\omega_g \underline{\underline{c}} - \dfrac{\omega_g^2}{\omega+i\omega_c} \hat{z}\,\hat{z}}{(\omega + i\omega_c)^2 - \omega_g^2} \cdot \underline{E} \tag{III.B.34}$$

where ω_p is the plasma frequency of the free electrons defined as

$$\omega_p^2 \equiv \frac{nq^2}{m\varepsilon_o} \tag{III.B.35}$$

The equation of the convection current density can be put into a form similar to that for the conduction current density,

$$\underline{j}_{conv} = \underline{\underline{\tau}} \cdot \underline{E} \tag{III.B.36}$$

if

$$\underline{\underline{\tau}} \equiv i\omega\varepsilon_o \frac{\omega_p^2}{\omega} \frac{(\omega + i\omega_c)\underline{\underline{u}} - i\omega_g \underline{\underline{c}} - \frac{\omega_g^2}{\omega + i\omega_c} \hat{z}\,\hat{z}}{(\omega + i\omega_c)^2 - \omega_g^2} \tag{III.B.37}$$

3. Constitutive Parameters

In a conducting dielectric

$$\underline{j}_o = \underline{j}_{cond} = \underline{\underline{\sigma}} \cdot \underline{E} \tag{III.B.38}$$

and

$$\underline{\underline{\sigma}} = \underline{\underline{u}}\,\sigma \tag{III.B.39}$$

$$\underline{\underline{\zeta}} = \underline{\underline{u}}\,\varepsilon_r \tag{III.B.40}$$

$$\underline{\underline{\eta}} = \underline{\underline{u}}\,\mu_r \tag{III.B.41}$$

Substituting these parameters into Maxwell's equations yields

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o\mu_r \, \underline{H} \tag{III.B.42}$$

$$\underline{\nabla} \wedge \underline{H} = -i\omega\varepsilon_o(\varepsilon_r + i \, \frac{\sigma}{\omega\varepsilon_o}) \, \underline{E} \tag{III.B.43}$$

To put Eqs. III.B.42 and III.B.43 into a symmetric form, the relative permittivity and permeability dyads are redefined as

$$\underline{\underline{\zeta}} = \underline{\underline{u}} \, \zeta_o \qquad \zeta_o \equiv \varepsilon_r + i \, \frac{\sigma}{\omega\varepsilon_o} \tag{III.B.44}$$

$$\underline{\underline{\eta}} = \underline{\underline{u}} \, \eta_o \qquad \eta_o \equiv \mu_r \tag{III.B.45}$$

so that

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o\eta_o \, \underline{H} \tag{III.B.46}$$

$$\underline{\nabla} \wedge \underline{H} = -i\omega\varepsilon_o\zeta_o \, \underline{E} \tag{III.B.47}$$

In an anisotropic plasma

$$\underline{j} = \underline{j}_{conv} = \underline{\underline{\tau}} \cdot \underline{E} \tag{III.B.48}$$

and

$$\underline{\underline{\tau}} = i\omega\varepsilon_o \, \frac{\omega_p^2}{\omega} \, \frac{(\omega + i\omega_c)\underline{\underline{u}} - i\omega_g\underline{\underline{c}} - \dfrac{\omega_g^2}{\omega + i\omega_c} \, \hat{z} \, \hat{z}}{(\omega + i\omega_c)^2 - \omega_g^2} \tag{III.B.49}$$

$$\underline{\underline{\zeta}} = \underline{\underline{u}} \tag{III.B.50}$$

$$\underline{\underline{\eta}} = \underline{\underline{u}} \tag{III.B.51}$$

Substituting these parameters into Maxwell's equations yields

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o \underline{H} \tag{III.B.52}$$

$$\underline{\nabla} \wedge \underline{H} = -i\omega\varepsilon_o \left[\underline{\underline{u}} - \frac{\omega_p^2}{\omega} \frac{(\omega + i\omega_c)\underline{\underline{u}} - i\omega_g \underline{\underline{c}} - \dfrac{\omega_g^2}{\omega + i\omega_c} \hat{z}\,\hat{z}}{(\omega + i\omega_c)^2 - \omega_g^2} \right] \cdot \underline{E} \tag{III.B.53}$$

To put Eqs. III.B.52 and III.B.53 into a symmetric form, the relative permittivity and permeability dyads are redefined as

$$\underline{\underline{\zeta}} = \begin{cases} \zeta_\perp \underline{\underline{u}} + i\zeta_+ \underline{\underline{c}} + \zeta_x \hat{z}\,\hat{z} \\ \zeta_\perp \underline{\underline{t}} + i\zeta_+ \underline{\underline{c}} + \zeta_\parallel \hat{z}\,\hat{z} \end{cases} \tag{III.B.54}$$

$$\underline{\underline{\eta}} = \underline{\underline{u}} \tag{III.B.55}$$

where

$$\zeta_\perp \equiv 1 - \frac{\omega_p^2}{\omega} \frac{(\omega + i\omega_c)}{(\omega + i\omega_c)^2 - \omega_g^2} \tag{III.B.56}$$

$$\zeta_+ \equiv \frac{\omega_p^2}{\omega} \frac{\omega_g}{(\omega + i\omega_c)^2 - \omega_g^2} \tag{III.B.57}$$

$$\zeta_x \equiv \frac{\omega_p^2}{\omega} \frac{\omega_g^2}{\omega + i\omega_c} \frac{1}{(\omega + i\omega_c)^2 - \omega_g^2} \tag{III.B.58}$$

$$\zeta_\parallel \equiv 1 - \frac{\omega_p^2}{\omega} \frac{1}{\omega + i\omega_c} \tag{III.B.59}$$

$$\left.\begin{matrix} \text{(III.B.58)} \\ \text{(III.B.59)} \end{matrix}\right\} \zeta_\parallel = \zeta_\perp + \zeta_x$$

and $\underline{\underline{t}}$ is the transverse part of the unit dyad.

Then

$$\underline{\nabla} \wedge \underline{E} = i\omega\mu_o \, \underline{\underline{\eta}} \cdot \underline{H} \qquad \text{(III.B.60)}$$

$$\underline{\nabla} \wedge \underline{H} = -i\omega\varepsilon_o \, \underline{\underline{\zeta}} \cdot \underline{E} \qquad \text{(III.B.61)}$$

If $\omega_g \to 0$, Eqs. III.B.56-59 reduce to the isotropic case

$$\lim_{\omega_g \to 0} \underline{\underline{\zeta}} = \left(1 - \frac{\omega_p^2}{\omega} \frac{1}{\omega + i\omega_c}\right) \underline{\underline{u}} \qquad \text{(III.B.62)}$$

$$\lim_{\omega_g \to 0} \underline{\underline{\eta}} = \underline{\underline{u}} \qquad \text{(III.B.63)}$$

4. Potentials

(a) Dielectric. Maxwell's equations in a stationary dielectric can be written in the alternate form

$$\underline{\nabla} \wedge \underline{E} = -\frac{k_o^2 \, \kappa^2}{i\omega\varepsilon_o\zeta_o} \, \underline{H} \qquad \text{(III.B.64)}$$

$$\underline{\nabla} \wedge \underline{H} = \underline{j}_o + \frac{k_o^2 \, \kappa^2}{i\omega\mu_o\eta_o} \, \underline{E} \qquad \text{(III.B.65)}$$

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho_o}{\varepsilon_o \, \varepsilon_r} \qquad \text{(III.B.66)}$$

$$\underline{\nabla} \cdot \underline{H} = 0 \qquad \text{(III.B.67)}$$

where

$$\kappa^2 \equiv \zeta_o\eta_o = \mu_r\left(\varepsilon_r + i\,\frac{\sigma}{\omega\varepsilon_o}\right) \qquad \text{(III.B.68)}$$

and

$$k_o \equiv \omega/c \qquad \text{(III.B.69)}$$

The form of Eqs. III.B.64-67 suggest that an electric vector potential $\underline{\Pi}_e$ can be introduced by letting

$$\underline{H} \equiv -i\omega\varepsilon_o\zeta_o \, \underline{\nabla} \wedge \underline{\Pi}_e \qquad \text{(III.B.70)}$$

$$\underline{E} \equiv \underline{\nabla}\,\underline{\nabla} \cdot \underline{\Pi}_e + k_o^2\kappa^2 \, \underline{\Pi}_e \qquad \text{(III.B.71)}$$

When $\underline{E}$ and $\underline{H}$ are substituted into Eqs. III.B.64 and III.B.65, the following vector partial differential equation is obtained, relating the electric vector potential to the source term

$$(\nabla^2 + k_o^2\kappa^2)\,\underline{\Pi}_e = \frac{\underline{j}_o}{i\omega\varepsilon_o\zeta_o} \qquad \text{(III.B.72)}$$

The relationship between $\underline{E}$ and $\underline{\Pi}_e$ can then be reduced to the simpler form of

$$\underline{E} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\Pi}_e + \frac{\underline{j}_o}{i\omega\varepsilon_o\zeta_o} \qquad \text{(III.B.73)}$$

or, in regions where $\underline{j}_o = 0$, to

$$\underline{E} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\Pi}_e \qquad \text{(III.B.74)}$$

To the solutions for $\underline{E}$ and $\underline{H}$ can be added any solution of the homogeneous Maxwell equations. In regions free of charge and current Maxwell's equations reduce to the expressions

$$\underline{\nabla} \wedge \underline{E} = -\frac{k_o^2\kappa^2}{i\omega\varepsilon_o\zeta_o}\,\underline{H} \qquad \text{(III.B.75)}$$

$$\underline{\nabla} \wedge \underline{H} = \frac{k_o^2 \kappa^2}{i\omega\mu_o \eta_o} \underline{E} \qquad \text{(III.B.76)}$$

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \text{(III.B.77)}$$

$$\underline{\nabla} \cdot \underline{H} = 0 \qquad \text{(III.B.78)}$$

The form of Eqs. III.B.75-78 suggest that a magnetic vector potential $\underline{\Pi}_m$ can be introduced by letting

$$\underline{E} \equiv i\omega\mu_o \eta_o \, \underline{\nabla} \wedge \underline{\Pi}_m \qquad \text{(III.B.79)}$$

$$\underline{H} \equiv \underline{\nabla}\,\underline{\nabla} \cdot \underline{\Pi}_m + k_o^2 \kappa^2 \, \underline{\Pi}_m \qquad \text{(III.B.80)}$$

When $\underline{E}$ and $\underline{H}$ are substituted into Eqs. III.B.75 and III.B.76, the following vector partial differential equation is found for the magnetic vector potential:

$$(\nabla^2 + k_o^2 \kappa^2) \, \underline{\Pi}_m = 0 \qquad \text{(III.B.81)}$$

The relationship between $\underline{H}$ and $\underline{\Pi}_m$ can then be reduced to the simpler form of

$$\underline{H} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\Pi}_m \qquad \text{(III.B.82)}$$

In summary, the most general solution of Maxwell's equations in a stationary dielectric are

$$\underline{E} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\Pi}_e + i\omega\mu_o\eta_o \, \underline{\nabla} \wedge \underline{\Pi}_m \tag{III.B.83}$$

$$\underline{H} = \underline{\nabla} \wedge \underline{\nabla} \wedge \underline{\Pi}_m - i\omega\varepsilon_o\zeta_o \, \underline{\nabla} \wedge \underline{\Pi}_e \tag{III.B.84}$$

where

$$(\nabla^2 + k_o^2\kappa^2) \, \underline{\Pi}_e = \frac{\underline{j}_o}{i\omega\varepsilon_o\zeta_o} \tag{III.B.85}$$

$$(\nabla^2 + k_o^2\kappa^2) \, \underline{\Pi}_m = 0 \tag{III.B.86}$$

For the cylindrical system of this study, in which axial and transverse directions can be identified, it is convenient to obtain expressions relating the axial and transverse components of the field vectors to the axial and transverse components of the vector potentials. This can be done by expression the field vectors in the following form:

$$\underline{\upsilon}(t,\rho,\phi,z) = \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, \underline{v}(\omega,\rho,\phi,k_o\gamma) e^{ik_o\gamma(z-z_s)} e^{-i\omega t} \tag{III.B.87}$$

$$\underline{v}(\omega,\rho,\phi,k_o\gamma) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \, \underline{\upsilon}(t,\rho,\phi,z) \, e^{-ik_o\gamma(z-z_s)} e^{i\omega t} \tag{III.B.88}$$

which is just a double Fourier integral transformation between the variables t, ω, z, and $k_o\gamma$.

Again, let $\underline{v}$ represent any transformed vector; but now $\underline{v} \equiv \underline{v}\,(\omega,\rho,\emptyset,k_o\gamma)$. Also, let any vector $\underline{v}$ be separated into the axial and transverse components v_z and $\underline{v}_t$, respectively. The curl, divergence, and gradient operations must also be separated into axial and transverse components. Consequently, Eqs. III.B.83 and III.B.84 as re-expressed in terms of the axial and transverse components of the transformed variables become

$$\underline{E}^t + \hat{z}\,E^z = -\hat{z}[(\nabla^t)^2\Pi^z_e - ik_o\gamma\,\underline{\nabla}^t\cdot\underline{\Pi}^t_e] + \underline{\underline{c}}\cdot\underline{\nabla}^t(\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_e) + ik_o\gamma\,\underline{\nabla}^t\Pi^z_e$$
$$+\,k_o^2\gamma^2\underline{\Pi}^t_e + i\omega\mu_o\eta_o[-\hat{z}\,\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_m - \underline{\underline{c}}\cdot(\underline{\nabla}^t\Pi^z_m - ik_o\gamma\,\underline{\Pi}^t_m)] \qquad \text{(III.B.89)}$$

$$\underline{H}^t + \hat{z}\,H^z = -\hat{z}[(\nabla^t)^2\Pi^z_m - ik_o\gamma\underline{\nabla}^t\cdot\underline{\Pi}^t_m] + \underline{\underline{c}}\cdot\underline{\nabla}^t(\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_m) + ik_o\gamma\,\underline{\nabla}^t\Pi^z_m$$
$$+\,k_o^2\gamma^2\,\underline{\Pi}^t_m - i\omega\varepsilon_o\zeta_o[-\hat{z}\,\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_e - \underline{\underline{c}}\cdot(\underline{\nabla}^t\Pi^z_e - ik_o\gamma\,\underline{\Pi}^t_e)] \qquad \text{(III.B.90)}$$

Again, the various vector operations have been expressed in terms of the dyad $\underline{\underline{c}}$.

The axial and transverse components of the electric field vector as derived from Eq. III.B. 89 are

$$E^z = -[(\nabla^t)^2\Pi^z_e - ik_o\gamma\,\underline{\nabla}^t\cdot\underline{\Pi}^t_e] - i\omega\mu_o\eta_o\,\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_m \qquad \text{(III.B.91)}$$

$$\underline{E}^t = \underline{\underline{c}}\cdot\underline{\nabla}^t(\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\Pi}^t_e) + ik_o\gamma\,\underline{\nabla}^t\Pi^z_e + k_o^2\gamma^2\,\underline{\Pi}^t_e$$
$$- i\omega\mu_o\eta_o\,\underline{\underline{c}}\cdot(\underline{\nabla}^t\Pi^z_m - ik_o\gamma\,\underline{\Pi}^t_m) \qquad \text{(III.B.92)}$$

Similarly, the axial and transverse components of the magnetic field vector as derived from Eq. III.B.90 are

$$H^z = -[(\nabla^t)^2\Pi^z_m - ik_o\gamma\, \underline{\nabla}^t \cdot \underline{\Pi}^t_m] + i\omega\varepsilon_o\zeta_o\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\Pi}^t_e \qquad \text{(III.B.93)}$$

$$\underline{H}^t = \underline{\underline{c}} \cdot \underline{\nabla}^t(\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\Pi}^t_m) + ik_o\gamma\, \underline{\nabla}^t\Pi^z_m + k_o^2\gamma^2\underline{\Pi}^t_m$$

$$+ i\omega\varepsilon_o\zeta_o\, \underline{\underline{c}} \cdot (\underline{\nabla}^t\Pi^z_e - ik_o\gamma\, \underline{\Pi}^t_e) \qquad \text{(III.B.94)}$$

By a similar procedure, the vector partial differential equations satisfied by the electric and magnetic vector potentials are

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\, \underline{\Pi}_e = \frac{\underline{j}_o}{i\omega\varepsilon_o\zeta_o} \qquad \text{(III.B.95)}$$

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\, \underline{\Pi}_m = 0 \qquad \text{(III.B.96)}$$

where the transverse wave number κ_t is defined as

$$\kappa_t^2 \equiv \kappa^2 - \gamma^2 \qquad \text{(III.B.97)}$$

In regions where the source currents are zero, the only nonzero components of the vector potentials are the axial components of $\underline{\Pi}_e$ and $\underline{\Pi}_m$. That the problem can be scalarized with just two nonzero components of the vector potentials is shown in the following section of this study. For this case the axial and transverse components of the electric field vectors reduce to the expressions

$$E^z = -(\nabla^t)^2 \Pi_e^z \tag{III.B.98}$$

$$\underline{E}^t = ik_o\gamma \underline{\nabla}^t \Pi_e^z - i\omega\mu_o\eta_o \underline{\underline{c}} \cdot \underline{\nabla}^t \Pi_m^z \tag{III.B.99}$$

Similarly, the axial and transverse components of the magnetic field vector reduce to the expressions

$$H^z = -(\nabla^t)^2 \Pi_m^z \tag{III.B.100}$$

$$\underline{H}^t = ik_o\gamma \underline{\nabla}^t \Pi_m^z + i\omega\varepsilon_o\zeta_o \underline{\underline{c}} \cdot \underline{\nabla}^t \Pi_e^z \tag{III.B.101}$$

The two nontrivial equations for the nonzero components of the vector potentials are, of course,

$$[(\nabla^t)^2 + k_o^2\kappa_t^2] \Pi_e^z = 0 \tag{III.B.102}$$

$$[(\nabla^t)^2 + k_o^2\kappa_t^2] \Pi_m^z = 0 \tag{III.B.103}$$

The cylindrical components of the field vectors are

$$\begin{aligned} E^\rho &= ik_o\gamma \frac{\partial \Pi_e^z}{\partial \rho} + i\omega\mu_o\eta_o \frac{1}{\rho} \frac{\partial \Pi_m^z}{\partial \phi} \\ E^\phi &= ik_o\gamma \frac{1}{\rho} \frac{\partial \Pi_e^z}{\partial \phi} - i\omega\mu_o\eta_o \frac{\partial \Pi_m^z}{\partial \rho} \\ E^z &= k_o^2\kappa_t^2 \Pi_e^z \end{aligned} \tag{III.B.104}$$

and

$$H^{\rho} = ik_o\gamma \frac{\partial \Pi_m^z}{\partial \rho} - i\omega\varepsilon_o\zeta_o \frac{1}{\rho} \frac{\partial \Pi_e^z}{\partial \phi}$$

$$H^{\phi} = ik_o\gamma \frac{1}{\rho} \frac{\partial \Pi_m^z}{\partial \phi} + i\omega\varepsilon_o\zeta_o \frac{\partial \Pi_e^z}{\partial \rho} \qquad \text{(III.B.105)}$$

$$H^z = k_o^2\kappa_t^2 \Pi_m^z$$

In regions where the source currents are not zero, more than two nonzero components of the vector potentials must exist. For a general source distribution, it must be assumed that all of the components of the vector potentials are nonzero. The scalarization of the problem for the special case of a horizontal dipole will be considered later in this study.

For the general case, the cylindrical components of the field vectors are

$$E^{\rho} = -\frac{1}{\rho}\frac{\partial}{\partial \phi}\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \Pi_e^{\phi}) + \frac{1}{\rho}\frac{\partial \Pi_e^{\rho}}{\partial \phi}\right] + ik_o\gamma \frac{\partial \Pi_e^z}{\partial \rho} + k_o^2\gamma^2 \Pi_e^{\rho}$$

$$- i\omega\mu_o\eta_o\left(-\frac{1}{\rho}\frac{\partial \Pi_m^z}{\partial \phi} + ik_o\gamma \Pi_m^{\phi}\right)$$

$$E^{\phi} = \frac{\partial}{\partial \rho}\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \Pi_e^{\phi}) + \frac{1}{\rho}\frac{\partial \Pi_e^{\rho}}{\partial \phi}\right] + ik_o\gamma \frac{1}{\rho}\frac{\partial \Pi_e^z}{\partial \phi} + k_o^2\gamma^2 \Pi_e^{\phi}$$

$$- i\omega\mu_o\eta_o\left(\frac{\partial \Pi_m^z}{\partial \rho} - ik_o\gamma \Pi_m^{\rho}\right) \qquad \text{(III.B.106)}$$

$$E^z = k_o^2\kappa_t^2 \Pi_e^z + ik_o\gamma\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \Pi_e^{\rho}) + \frac{1}{\rho}\frac{\partial \Pi_e^{\phi}}{\partial \phi}\right]$$

$$- i\omega\mu_o\eta_o\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \Pi_m^{\phi}) + \frac{1}{\rho}\frac{\partial \Pi_m^{\rho}}{\partial \phi}\right]$$

and

$$H^{\rho} = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Pi_m^{\phi}) + \frac{1}{\rho}\frac{\partial\Pi_m^{\rho}}{\partial\phi}\right] + ik_o\gamma\frac{\partial\Pi_m^z}{\partial\rho} + k_o^2\gamma^2\Pi_m^{\rho}$$

$$+ i\omega\varepsilon_o\zeta_o\left(-\frac{1}{\rho}\frac{\partial\Pi_e^z}{\partial\phi} + ik_o\gamma\,\Pi_e^{\phi}\right)$$

$$H^{\phi} = \frac{\partial}{\partial\rho}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Pi_m^{\phi}) + \frac{1}{\rho}\frac{\partial\Pi_m^{\rho}}{\partial\phi}\right] + ik_o\gamma\frac{1}{\rho}\frac{\partial\Pi_m^z}{\partial\phi} + k_o^2\gamma^2\Pi_m^{\phi} \qquad \text{(III.B.107)}$$

$$+ i\omega\varepsilon_o\zeta_o\left(\frac{\partial\Pi_e^z}{\partial\rho} - ik_o\gamma\,\Pi_e^{\rho}\right)$$

$$H^z = k_o^2\kappa_t^2\Pi_m^z + ik_o\gamma\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Pi_m^{\rho}) + \frac{1}{\rho}\frac{\partial\Pi_m^{\phi}}{\partial\phi}\right] + i\omega\varepsilon_o\zeta_o\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Pi_e^{\phi}) + \frac{1}{\rho}\frac{\partial\Pi_e^{\rho}}{\partial\phi}\right]$$

Later in the study, the rectangular components of the potentials will be developed. Therefore, the cylindrical components of the field vectors will be developed in terms of the rectangular components of the vector potentials by substituting

$$\Pi^{\rho} = \Pi^x \cos\phi + \Pi^y \sin\phi \qquad \text{(III.B.108)}$$

$$\Pi^{\phi} = -\Pi^x \sin\phi + \Pi^y \cos\phi \qquad \text{(III.B.109)}$$

into Eqs. III.B.106 and III.B.107 . When this substitution is carried out, the following results for the field vectors are obtained:

$$E^{\rho} = ik_o\gamma \frac{\partial \Pi_e^z}{\partial \rho} + i\omega\mu_o\eta_o \frac{1}{\rho} \frac{\partial \Pi_m^z}{\partial \phi}$$

$$+ [-\frac{1}{\rho^2}(-\frac{\partial \Pi_e^x}{\partial \phi} + \frac{\partial^2 \Pi_e^y}{\partial \phi^2}) + \frac{1}{\rho}(-\frac{\partial^2 \Pi_e^x}{\partial \rho \partial \phi} - \frac{\partial \Pi_e^y}{\partial \rho}) + k_o^2\gamma^2\Pi_e^y - \omega\mu_o\eta_o k_o\gamma\, \Pi_m^x]\sin\phi$$

$$+ [-\frac{1}{\rho^2}(\frac{\partial^2 \Pi_e^x}{\partial \phi^2} + \frac{\partial \Pi_e^y}{\partial \phi}) + \frac{1}{\rho}(-\frac{\partial \Pi_e^x}{\partial \rho} + \frac{\partial^2 \Pi_e^y}{\partial \rho \partial \phi}) + k_o^2\gamma^2\Pi_e^x + \omega\mu_o\eta_o k_o\gamma\, \Pi_m^y] \cos\phi$$

$$E^{\phi} = ik_o\gamma \frac{1}{\rho} \frac{\partial \Pi_e^z}{\partial \phi} - i\omega\mu_o\eta_o \frac{\partial \Pi_m^z}{\partial \rho}$$

$$+ (-\frac{1}{\rho^2} \frac{\partial \Pi_e^y}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 \Pi_e^y}{\partial \rho \partial \phi} + \frac{\partial^2 \Pi_e^x}{\partial \rho^2} - k_o^2\gamma^2\Pi_e^x - \omega\mu_o\eta_o k_o\gamma\, \Pi_m^y) \sin\phi$$

$$+ (-\frac{1}{\rho^2} \frac{\partial \Pi_e^x}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 \Pi_e^x}{\partial \rho \partial \phi} - \frac{\partial^2 \Pi_e^y}{\partial \rho^2} + k_o^2\gamma^2\Pi_e^y - \omega\mu_o\eta_o k_o\gamma\, \Pi_m^x) \cos\phi$$

$$E^z = k_o^2\kappa_t^2\Pi_e^z$$

$$+ [ik_o\gamma(-\frac{1}{\rho} \frac{\partial \Pi_e^x}{\partial \phi} + \frac{\partial \Pi_e^y}{\partial \rho}) + i\omega\mu_o\eta_o(-\frac{1}{\rho} \frac{\partial \Pi_m^y}{\partial \phi} - \frac{\partial \Pi_m^x}{\partial \rho})] \sin\phi$$

$$+ [ik_o\gamma(\frac{1}{\rho} \frac{\partial \Pi_e^y}{\partial \phi} + \frac{\partial \Pi_e^x}{\partial \rho}) + i\omega\mu_o\eta_o(-\frac{1}{\rho} \frac{\partial \Pi_m^x}{\partial \phi} + \frac{\partial \Pi_m^y}{\partial \rho})] \cos\phi \qquad \text{(III.B.110)}$$

and

$$H^{\rho} = ik_o\gamma \frac{\partial \Pi_m^z}{\partial \rho} - i\omega\varepsilon_o\zeta_o \frac{1}{\rho} \frac{\partial \Pi_e^z}{\partial \phi}$$

$$+ [-\frac{1}{\rho^2}(-\frac{\partial \Pi_m^x}{\partial \phi} + \frac{\partial^2 \Pi_m^y}{\partial \phi^2}) + \frac{1}{\rho}(-\frac{\partial^2 \Pi_m^x}{\partial \rho \partial \phi} + \frac{\partial \Pi_m^y}{\partial \rho}) + k_o^2\gamma^2\Pi_m^y + \omega\varepsilon_o\zeta_o k_o\gamma\, \Pi_e^x]\sin\phi$$

$$+ [-\frac{1}{\rho^2}(\frac{\partial^2 \Pi_m^x}{\partial \phi^2} + \frac{\partial \Pi_m^y}{\partial \phi}) + \frac{1}{\rho}(-\frac{\partial \Pi_m^x}{\partial \rho} + \frac{\partial^2 \Pi_m^y}{\partial \rho \partial \phi}) + k_o^2\gamma^2\Pi_m^x - \omega\varepsilon_o\zeta_o k_o\gamma\, \Pi_e^y] \cos\phi$$

$$H^{\phi} = ik_o\gamma \frac{1}{\rho}\frac{\partial \Pi^z_m}{\partial \phi} + i\omega\varepsilon_o\zeta_o \frac{\partial \Pi^z_e}{\partial \rho}$$

$$+(-\frac{1}{\rho^2}\frac{\partial \Pi^y_m}{\partial \phi} + \frac{1}{\rho}\frac{\partial^2 \Pi^y_m}{\partial \rho \partial \phi} + \frac{\partial^2 \Pi^x_m}{\partial \rho^2} - k_o^2\gamma^2\Pi^x_m + \omega\varepsilon_o\zeta_o\, k_o\gamma\, \Pi^y_e) \sin \phi$$

$$+(-\frac{1}{\rho^2}\frac{\partial \Pi^x_m}{\partial \phi} + \frac{1}{\rho}\frac{\partial^2 \Pi^x_m}{\partial \rho \partial \phi} - \frac{\partial^2 \Pi^y_m}{\partial \rho^2} + k_o^2\gamma^2\Pi^y_m + \omega\varepsilon_o\zeta_o k_o\gamma\, \Pi^x_e) \cos \phi$$

$$H^z = k_o^2\kappa_t^2\, \Pi^z_m \tag{III.B.111}$$

$$+ [ik_o\gamma(-\frac{1}{\rho}\frac{\partial \Pi^x_m}{\partial \phi} + \frac{\partial \Pi^y_m}{\partial \rho}) - i\omega\varepsilon_o\zeta_o\, (-\frac{1}{\rho}\frac{\partial \Pi^y_e}{\partial \phi} - \frac{\partial \Pi^x_e}{\partial \rho})] \sin \phi$$

$$+ [ik_o\gamma(\frac{1}{\rho}\frac{\partial \Pi^y_m}{\partial \phi} + \frac{\partial \Pi^x_m}{\partial \rho}) - i\omega\varepsilon_o\zeta_o\, (-\frac{1}{\rho}\frac{\partial \Pi^x_e}{\partial \phi} + \frac{\partial \Pi^y_e}{\partial \rho})] \cos \phi$$

(b) Alternative method. An alternative method of deriving the field relationships in a stationary dielectric is now presented since this method can be generalized to solve the case in which the medium is anisotropic. Having worked the simpler isotropic case first will make the effects of the anisotropy more discernable. Also, the scalarization of the problem is easily handled by this method.

Maxwell's equations in a stationary dielectric can be written in terms of the axial and transverse components of the field vectors as

$$(\underline{\nabla}^t + \hat{z}\, ik_o\gamma) \wedge (\underline{E}^t + \hat{z}\, E^z) = i\omega\mu_o\eta_o(\underline{H}^t + \hat{z}\, H^z) \tag{III.B.112}$$

$$(\underline{\nabla}^t + \hat{z}\, ik_o\gamma) \wedge (\underline{H}^t + \hat{z}\, H^z) = -i\omega\varepsilon_o\zeta_o(\underline{E}^t + \hat{z}\, E^z) \tag{III.B.113}$$

When the curl operations are expanded in terms of the dyad $\underline{\underline{c}}$ as described in the appendix, the axial and transverse components of Eqs. III.B.112 and III.B.113 can be separated as follows:

$$\left.\begin{aligned} -\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t &= i\omega\mu_o\eta_o\, H^z \\ -\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t &= -i\omega\varepsilon_o\zeta_o\, E^z \end{aligned}\right\} \text{axial} \qquad \text{(III.B.114)}$$

$$\left.\begin{aligned} -\underline{\underline{c}} \cdot (\underline{\nabla}^t E^z - ik_o\gamma\ \underline{E}^t) &= i\omega\mu_o\eta_o\, \underline{H}^t \\ -\underline{\underline{c}} \cdot (\underline{\nabla}^t H^z - ik_o\gamma\ \underline{H}^t) &= -i\omega\varepsilon_o\zeta_o\, \underline{E}^t \end{aligned}\right\} \text{transverse} \qquad \text{(III.B.115)}$$

After being premultiplied from the left by the dyad $\underline{\underline{c}}$ and rearranged, the transverse equations become

$$ik_o\gamma\ \underline{\underline{t}} \cdot \underline{E}^t + i\omega\mu_o\eta_o\ \underline{\underline{c}} \cdot \underline{H}^t = \underline{\underline{t}} \cdot \underline{\nabla}^t\, E^z \qquad \text{(III.B.116)}$$

$$-i\omega\varepsilon_o\zeta_o\ \underline{\underline{c}} \cdot \underline{E}^t + ik_o\gamma\ \underline{\underline{t}} \cdot \underline{H}^t = \underline{\underline{t}} \cdot \underline{\nabla}^t\, H^z \qquad \text{(III.B.117)}$$

The transverse field components in terms of the axial field components are obtained by solving simultaneously Eqs. III.B.116 and III.B.117. The resulting expressions are

$$k_o^2\kappa_t^2\ \underline{\underline{t}} \cdot \underline{E}^t = ik_o\gamma\ \underline{\underline{t}} \cdot \underline{\nabla}^t\, E^z - i\omega\mu_o\eta_o\ \underline{\underline{c}} \cdot \underline{\nabla}^t\, H^z \qquad \text{(III.B.118)}$$

$$k_o^2\kappa_t^2\ \underline{\underline{t}} \cdot \underline{H}^t = ik_o\gamma\ \underline{\underline{t}} \cdot \underline{\nabla}^t\, H^z + i\omega\varepsilon_o\zeta_o\ \underline{\underline{c}} \cdot \underline{\nabla}^t\, E^z \qquad \text{(III.B.119)}$$

Operating on the transverse equations with $\underline{\nabla}^t\cdot$ one finds that

$$-\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t E^z + ik_o\gamma\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t = i\omega\mu_o\eta_o\, \underline{\underline{t}} \cdot \underline{\nabla}^t \cdot \underline{H}^t \quad \text{(III.B.120)}$$

$$-\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t H^z + ik_o\gamma\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t = -i\omega\varepsilon_o\zeta_o\, \underline{\underline{t}} \cdot \underline{\nabla}^t \cdot \underline{E}^t \quad \text{(III.B.121)}$$

As shown in the appendix, $\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t$ operating on any vector is zero. Therefore, with the aid of the axial equations, Eqs. III.B.120 and III.B.121 become

$$\underline{\nabla}^t \cdot \underline{H}^t = -ik_o\gamma\, H^z \quad \text{(III.B.122)}$$

$$\underline{\nabla}^t \cdot \underline{E}^t = -ik_o\gamma\, E^z \quad \text{(III.B.123)}$$

Operating on the transverse equations with $\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot$, one finds that

$$-\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t E^z + ik_o\gamma\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t = i\omega\mu_o\eta_o\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{H}^t \quad \text{(III.B.124)}$$

$$-\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t H^z + ik_o\gamma\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t = -i\omega\varepsilon_o\zeta_o\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{E}^t \quad \text{(III.B.125)}$$

As shown in the appendix, $-\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot$ operating on any vector is the transverse divergence of that vector, and $\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot$ operating on any vector is equivalent to $\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot$ operating on that vector. Therefore, Eqs. III.B.124 and III.B.125 become

$$\underline{\nabla}^t \cdot \underline{\nabla}^t E^z - ik_o\gamma\, \underline{\nabla}^t \cdot \underline{E}^t = i\omega\mu_o\eta_o\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t \quad \text{(III.B.126)}$$

$$\underline{\nabla}^t \cdot \underline{\nabla}^t H^z - ik_o\gamma\, \underline{\nabla}^t \cdot \underline{H}^t = -i\omega\varepsilon_o\zeta_o\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t \quad \text{(III.B.127)}$$

Inserting Eqs. III.B.122 and III.B.123 into Eqs. III.B.126 and III.B.127, with the aid of the axial equations, yields

$$[(\nabla^t)^2 + k_o^2\kappa^2]\, E^z = k_o^2\gamma^2 E^z \qquad (III.B.128)$$

$$[(\nabla^t)^2 + k_o^2\kappa^2]\, H^z = k_o^2\gamma^2 H^z \qquad (III.B.129)$$

or

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\, E^z = 0 \qquad (III.B.130)$$

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\, H^z = 0 \qquad (III.B.131)$$

If potentials are now introduced by letting

$$E^z \equiv k_o^2\kappa_t^2\, \Pi_e^z \qquad (III.B.132)$$

$$H^z \equiv k_o^2\kappa_t^2\, \Pi_m^z \qquad (III.B.133)$$

then the equations for the electric and magnetic field vectors reduce to the simpler form

$$\underline{E} = ik_o\gamma\, \underline{\underline{t}} \cdot \underline{\nabla}^t \Pi_e^z - i\omega\mu_o\eta_o\, \underline{\underline{c}} \cdot \underline{\nabla}^t \Pi_m^z + \hat{z}\, k_o^2\kappa_t^2\, \Pi_e^z \qquad (III.B.134)$$

$$\underline{H} = ik_o\gamma\, \underline{\underline{t}} \cdot \underline{\nabla}^t \Pi_m^z + i\omega\varepsilon_o\zeta_o\, \underline{\underline{c}} \cdot \underline{\nabla}^t \Pi_e^z + \hat{z}\, k_o^2\kappa_t^2\, \Pi_m^z \qquad (III.B.135)$$

The partial differential equations satisfied by the electric and magnetic potentials are found by substituting the defining equations for the potentials into Eqs. III.B.130 and III.B.131

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\ \Pi_e^z = 0 \qquad \text{(III.B.136)}$$

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\ \Pi_m^z = 0 \qquad \text{(III.B.137)}$$

The cylindrical components of the field vectors are

$$E^\rho = ik_o\gamma \frac{\partial \Pi_e^z}{\partial \rho} + i\omega\mu_o\eta_o \frac{1}{\rho}\frac{\partial \Pi_m^z}{\partial \phi}$$

$$E^\phi = ik_o\gamma \frac{1}{\rho}\frac{\partial \Pi_e^z}{\partial \phi} - i\omega\mu_o\eta_o \frac{\partial \Pi_m^z}{\partial \rho} \qquad \text{(III.B.138)}$$

$$E^z = k_o^2\kappa_t^2\ \Pi_e^z$$

and

$$H^\rho = ik_o\gamma \frac{\partial \Pi_m^z}{\partial \rho} - i\omega\varepsilon_o\zeta_o \frac{1}{\rho}\frac{\partial \Pi_e^z}{\partial \phi}$$

$$H^\phi = ik_o\gamma \frac{1}{\rho}\frac{\partial \Pi_m^z}{\partial \phi} + i\omega\varepsilon_o\zeta_o \frac{\partial \Pi_e^z}{\partial \rho} \qquad \text{(III.B.139)}$$

$$H^z = k_o^2\kappa_t^2\ \Pi_m^z$$

A comparison of Eqs. III.B.138 and III.B.139 for the cylindrical components of the field vectors with Eqs. III.B.104 and III.B.105 derived in the previous section for the same vectors reveals that they are identical.

<u>(c) Plasma.</u> Maxwell's equations in a stationary plasma can be written in terms of the axial and transverse components of the field vectors as

$$(\underline{\nabla}^t + \hat{z}\ ik_o\gamma) \wedge (\underline{E}^t + \hat{z}\ E^z) = i\omega\mu_o(\underline{H}^t + \hat{z}\ H^z) \qquad \text{(III.B.140)}$$

$$(\underline{\nabla}^t + \hat{z}\ ik_o\gamma) \wedge (\underline{H}^t + \hat{z}\ H^z) = -i\omega\varepsilon_o(-\zeta_\perp \underline{\underline{t}} + i\zeta_+ \underline{\underline{c}}) \cdot \underline{E}^t + \hat{z}\,\zeta_{\parallel}\ E^z \qquad \text{(III.B.141)}$$

When the curl operations are expanded in terms of the dyad $\underline{\underline{c}}$ as described in the appendix, the axial and transverse components of Eqs. III.B.140 and III.B.141 can be separated as follows:

$$\left.\begin{aligned} -\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t &= i\omega\mu_o\ H^z \\ -\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t &= -i\omega\varepsilon_o\ \zeta_{\parallel}\ E^z \end{aligned}\right\} \text{axial} \qquad \text{(III.B.142)}$$

$$\left.\begin{aligned} -\underline{\underline{c}} \cdot \underline{\nabla}^t\ E^z + ik_o\gamma\ \underline{\underline{c}} \cdot \underline{E}^t &= i\omega\mu_o\ \underline{\underline{t}} \cdot \underline{H}^t \\ -\underline{\underline{c}} \cdot \underline{\nabla}^t H^z + ik_o\gamma\ \underline{\underline{c}} \cdot \underline{H}^t &= -i\omega\varepsilon_o(\zeta_\perp \underline{\underline{t}} + i\zeta_+\ \underline{\underline{c}}) \cdot \underline{E}^t \end{aligned}\right\} \text{transverse} \qquad \text{(III.B.143)}$$

After being premultiplied from the left by the dyad $\underline{\underline{c}}$, the transverse equations become

$$\underline{\nabla}^t\ E^z - ik_o\gamma\ \underline{E}^t = i\omega\mu_o\ \underline{\underline{c}} \cdot \underline{H}^t \qquad \text{(III.B.144)}$$

$$\underline{\nabla}^t\ H^z - ik_o\gamma\ \underline{H}^t = -i\omega\varepsilon_o(\zeta_\perp\ \underline{\underline{c}} - i\zeta_+\ \underline{\underline{t}}) \cdot \underline{E}^t \qquad \text{(III.B.145)}$$

After some rearrangement, the transverse equations become

$$ik_o\gamma\ \underline{\underline{t}} \cdot \underline{E}^t + i\omega\mu_o\ \underline{\underline{c}} \cdot \underline{H}^t = \underline{\underline{t}} \cdot \underline{\nabla}^t\ E^z \qquad \text{(III.B.146)}$$

$$-i\omega\varepsilon_o(\zeta_\perp \underline{\underline{c}} - i\zeta_+ \underline{\underline{t}})\cdot\underline{E}^t + ik_o\gamma\,\underline{\underline{t}}\cdot\underline{H}^t = \underline{\underline{t}}\cdot\underline{\nabla}^t H^z \qquad \text{(III.B.147)}$$

The transverse field components in terms of the axial field components are obtained by solving simultaneously Eqs. III.B.146 and III.B.147. The resulting expressions are

$$k_o^2\,\Delta\underline{E}^t = \{ik_o\gamma(\zeta_\perp-\gamma^2)\underline{\underline{t}} + k_o\gamma\,\zeta_+\underline{\underline{c}}\}\cdot\underline{\nabla}^t E^z + \{\omega\mu_o\zeta_+\underline{\underline{t}} - i\omega\mu_o(\zeta_\perp-\gamma^2)\underline{\underline{c}}\}\cdot\underline{\nabla}^t H^z \qquad \text{(III.B.148)}$$

$$k_o^2\,\Delta\underline{H}^t = \{ik_o\gamma(\zeta_\perp-\gamma^2)\underline{\underline{t}} + k_o\gamma\,\zeta_+\underline{\underline{c}}\}\cdot\underline{\nabla}^t H^z + \{-\omega\varepsilon_o\zeta_+\gamma^2\underline{\underline{t}} + i\omega\varepsilon_o[(\zeta_\perp-\gamma^2)\zeta_\perp - \zeta_+^2]\,\underline{\underline{c}}\}\cdot\underline{\nabla}^t E^z \qquad \text{(III.B.149)}$$

where

$$\Delta \equiv (\zeta_\perp - \gamma^2)^2 + (i\zeta_+)^2 \qquad \text{(III.B.150)}$$

Operating on the transverse equations with $\underline{\nabla}^t\cdot$, one finds that

$$-\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\nabla}^t E^z + ik_o\gamma\quad \underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{E}^t = i\omega\mu_o\,\underline{\underline{t}}\cdot\underline{\nabla}^t\cdot\underline{H}^t \qquad \text{(III.B.151)}$$

$$-\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\nabla}^t H^z + ik_o\gamma\quad \underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{H}^t = -i\omega\varepsilon_o(\zeta_\perp\underline{\underline{t}}\cdot\underline{\nabla}^t\cdot\underline{E}^t + i\zeta_+\,\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{E}^t) \qquad \text{(III.B.152)}$$

As noted in the previous section, $\underline{\nabla}^t\cdot\underline{\underline{c}}\cdot\underline{\nabla}^t$ operating on any vector is zero. Therefore, with the aid of the axial equations, Eqs. III.B.151 and III.B.152 become

$$\underline{\nabla}^t\cdot\underline{H}^t = -\,ik_o\gamma\,H^z \qquad \text{(III.B.153)}$$

$$\zeta_\perp \underline{\nabla}^t \cdot \underline{E}^t = - ik_o\gamma\zeta_{\parallel} E^z - \omega\mu_o\zeta_+ H^z \qquad \text{(III.B.154)}$$

Operating on the transverse equations with $\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot$, one finds that

$$- \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t E^z + ik_o\gamma\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t = i\omega\mu_o\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{H}^t \qquad \text{(III.B.155)}$$

$$- \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{\nabla}^t H^z + ik_o\gamma\, \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t$$

$$= -i\omega\varepsilon_o(\zeta_\perp \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{E}^t + i\zeta_+ \underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t) \qquad \text{(III.B.156)}$$

As previously noted, $-\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot$ operating on any vector is the transverse divergence of that vector, and $\underline{\underline{c}} \cdot \underline{\nabla}^t \cdot$ operating on any vector is equivalent to $\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot$ operating on that vector. Therefore, the above equations become

$$\underline{\nabla}^t \cdot \underline{\nabla}^t E^z - ik_o\gamma\, \underline{\nabla}^t \cdot \underline{E}^t = i\omega\mu_o\, \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{H}^t \qquad \text{(III.B.157)}$$

$$\underline{\nabla}^t \cdot \underline{\nabla}^t H^z - ik_o\gamma\, \underline{\nabla}^t \cdot \underline{H}^t = -i\omega\varepsilon_o(\zeta_\perp \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{E}^t - i\zeta_+ \underline{\nabla}^t \cdot \underline{E}^t) \qquad \text{(III.B.158)}$$

Substituting Eqs. III.B.153 and III.B.154 into Eqs. III.B.157 and III.B.158, with the aid of the axial equations, yields

$$\left[(\nabla^t)^2 + k_o^2 \frac{\zeta_{\parallel}}{\zeta_\perp} (\zeta_\perp - \gamma^2)\right] E^z = -i\omega\mu_o k_o\gamma \frac{\zeta_+}{\zeta_\perp} H^z \qquad \text{(III.B.159)}$$

$$\left[(\nabla^t)^2 + k_o^2 \left(\frac{\zeta_\perp^2 - \zeta_+^2}{\zeta_\perp} - \gamma^2\right)\right] H^z = i\omega\varepsilon_o k_o\gamma \frac{\zeta_+\zeta_{\parallel}}{\zeta_\perp} E^z \qquad \text{(III.B.160)}$$

To simplify the notation in what follows, let

$$\zeta_e^2 \equiv \frac{\zeta_\parallel}{\zeta_\perp}(\zeta_\perp - \gamma^2) \tag{III.B.161}$$

$$\zeta_m^2 \equiv \frac{\zeta_\perp^2 - \zeta_+^2}{\zeta_\perp} - \gamma^2 \tag{III.B.162}$$

$$\eta_m \equiv -i\omega\mu_o k_o \gamma \frac{\zeta_+}{\zeta_\perp} \tag{III.B.163}$$

$$\eta_e \equiv i\omega\varepsilon_o k_o \gamma \frac{\zeta_+ \ \zeta_\parallel}{\zeta_\perp} \tag{III.B.164}$$

If potentials are now introduced by letting

$$E^z \equiv k_o^2 \frac{\Delta}{\zeta_\perp - \gamma^2} \Pi_e \tag{III.B.165}$$

$$H^z \equiv k_o^2 \frac{\Delta}{\zeta_\perp - \gamma^2} \Pi_m \tag{III.B.166}$$

then the equations for the electric and magnetic field vectors reduce to the simpler form

$$\underline{E}^t = [ik_o\gamma \ \underline{\underline{t}} + \frac{k_o\gamma\zeta_+}{\zeta_\perp - \gamma^2} \underline{\underline{c}}] \cdot \underline{\nabla}^t \Pi_e + [\frac{\omega\mu_o\zeta_+}{\zeta_\perp - \gamma^2} \underline{\underline{t}} - i\omega\mu_o \underline{\underline{c}}] \cdot \underline{\nabla}^t \Pi_m \tag{III.B.167}$$

$$\underline{H}^t = [ik_o\gamma \ \underline{\underline{t}} + \frac{k_o\gamma\zeta_+}{\zeta_\perp - \gamma^2} \underline{\underline{c}}] \cdot \underline{\nabla}^t \Pi_m + [-\frac{\omega\varepsilon_o\zeta_+\gamma^2}{\zeta_\perp - \gamma^2} \underline{\underline{t}} + i\omega\varepsilon_o \times (\zeta_\perp - \frac{\zeta_+^2}{\zeta_\perp - \gamma^2}) \underline{\underline{c}}] \cdot \underline{\nabla}^t \Pi_e \tag{III.B.168}$$

The partial differential equations satisfied by the electric and magnetic potentials are found by substituting the defining equations for the potentials into Eqs. III.B.159 and III.B.160

$$[(\nabla^t)^2 + k_o^2\zeta_e^2]\,\Pi_e = \eta_m\Pi_m \tag{III.B.169}$$

$$[(\nabla^t)^2 + k_o^2\zeta_m^2]\,\Pi_m = \eta_e\Pi_e \tag{III.B.170}$$

The cylindrical components of the field vectors are

$$\begin{aligned}
E^\rho &= ik_o\gamma\,\frac{\partial\Pi_e}{\partial\rho} - \frac{k_o\gamma\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{1}{\rho}\,\frac{\partial\Pi_e}{\partial\phi} + \frac{\omega\mu_o\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{\partial\Pi_m}{\partial\rho} + i\omega\mu_o\,\frac{1}{\rho}\,\frac{\partial\Pi_m}{\partial\phi} \\
E^\phi &= ik_o\gamma\,\frac{1}{\rho}\,\frac{\partial\Pi_e}{\partial\phi} + \frac{k_o\gamma\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{\partial\Pi_e}{\partial\rho} + \frac{\omega\mu_o\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{1}{\rho}\,\frac{\partial\Pi_m}{\partial\phi} - i\omega\mu_o\,\frac{\partial\Pi_m}{\partial\rho} \\
E^z &= k_o^2\,\frac{\Delta}{\zeta_\perp-\gamma^2}\,\Pi_e
\end{aligned} \tag{III.B.171}$$

and

$$\begin{aligned}
H^\rho &= ik_o\gamma\,\frac{\partial\Pi_m}{\partial\rho} - \frac{k_o\gamma\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{1}{\rho}\,\frac{\partial\Pi_m}{\partial\phi} - \frac{\omega\varepsilon_o\zeta_+\gamma^2}{\zeta_\perp-\gamma^2}\,\frac{\partial\Pi_e}{\partial\rho} - i\omega\varepsilon_o\left(\zeta_\perp - \frac{\zeta_+^2}{\zeta_\perp-\gamma^2}\right)\frac{1}{\rho}\,\frac{\partial\Pi_e}{\partial\phi} \\
H^\phi &= ik_o\gamma\,\frac{1}{\rho}\,\frac{\partial\Pi_m}{\partial\phi} + \frac{k_o\gamma\zeta_+}{\zeta_\perp-\gamma^2}\,\frac{\partial\Pi_m}{\partial\rho} - \frac{\omega\varepsilon_o\zeta_+\gamma^2}{\zeta_\perp-\gamma^2}\,\frac{1}{\rho}\,\frac{\partial\Pi_e}{\partial\phi} + i\omega\varepsilon_o\left(\zeta_\perp - \frac{\zeta_+^2}{\zeta_\perp-\gamma^2}\right)\frac{\partial\Pi_\varepsilon}{\partial\rho} \\
H^z &= k_o^2\,\frac{\Delta}{\zeta_\perp-\gamma^2}\,\Pi_m
\end{aligned} \tag{III.B.172}$$

5. Solutions of the Potential Equations

For a horizontal dipole located in the $\rho\phi$ plane as shown in Figure 21, the source distribution is represented by

$$\underline{j}_o = \hat{t}\, j_s \qquad \text{(III.B.173)}$$

where

$$\hat{t} \equiv \hat{x}t^x + \hat{y}t^y \qquad (t^x)^2 + (t^y)^2 = 1 \qquad \text{(III.B.174)}$$

and

$$j_s \equiv i_s l_s \frac{\delta(\rho-\rho_s)\ \delta(\phi-\phi_s)\ \delta(z-z_s)}{\rho} \qquad \text{(III.B.175)}$$

The transformed source distribution is

$$\mathrm{j}_s = \frac{i_s l_s}{\sqrt{2\pi}} \frac{\delta(\rho-\rho_s)\ \delta(\phi-\phi_s)}{\rho} \qquad \text{(III.B.176)}$$

To solve for the fields of a horizontal dipole in a stationary dielectric or in a stationary plasma, one must solve the equations

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\ \Pi_\infty = 0 \qquad \text{(III.B.177)}$$

$$[(\nabla^t)^2 + k_o^2\kappa_t^2]\ \Pi_o = \frac{i_s l_s/\sqrt{2\pi}}{i\omega\varepsilon_o\zeta_o} \frac{\delta(\rho-\rho_s)\ \delta(\phi-\phi_s)}{\rho} \qquad \text{(III.B.178)}$$

$$[(\nabla^t)^2 + k_o^2\zeta^2_{\{^e_m\}}]\Pi_{\{^e_m\}} = \eta_{\{^m_e\}}\Pi_{\{^m_e\}} \qquad \text{(III.B.179)}$$

where Π_o or Π_∞ represents any one of the three components of the electric or magnetic vector potentials.

In a cylindrical coordinate system

$$(\nabla^t)^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} \qquad \text{(III.B.180)}$$

Since the solution of each equation is periodic in ϕ with period 2π, it is convenient to expand the solution in the following form:

$$s(t,\rho,\phi,z) = \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d(k_o\gamma)}{\sqrt{2\pi}}\sum_{\nu=-\infty}^{\infty} s_\nu(\omega,\rho,k_o\gamma)e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}e^{-i\omega t}$$

(III.B.181)

$$s_\nu(\omega,\rho,k_o\gamma) = \int_{-\infty}^{\infty}\frac{dt}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\; s(t,\rho,\phi,z)$$

$$\times\, e^{-i\nu(\phi-\phi_s)}e^{-ik_o\gamma(z-z_s)}e^{i\omega t} \qquad \text{(III.B.182)}$$

The term ν is assumed to be an integer. In what follows, let any scalar s represent a function of t, ρ, ϕ, z and let s_ν represent the corresponding transformed scalar function of ω, ρ, $k_o\gamma$.

In terms of the transformed variables in the cylindrical coordinate system, Eq. III.B.177 becomes

$$\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + k_o^2\kappa_t^2\rho^2 - \nu^2\right]\Pi_\nu = 0 \qquad \text{(III.B.183)}$$

This equation is Bessel's equation of the complex argument $k_o\kappa_t\rho$ and integer order ν, i.e.,

$$\Pi_\nu = c_\nu^{(\pm)}\, Z_\nu^{(\pm)}\,(k_o\kappa_t\rho) \qquad \text{(III.B.184)}$$

The expression $c_\nu^{(\pm)} Z_\nu^{(\pm)}$ represents any linear combination of the independent pairs of Bessel, Neumann, or Hankel functions.

In terms of the transformed variables in the cylindrical coordinate system, Eq. III.B.178 becomes

$$\left[\rho \frac{\partial}{\partial\rho}\left(\rho \frac{\partial}{\partial\rho}\right)+ k_o^2\kappa_t^2\rho^2-\nu^2\right] \Pi_\nu = \frac{1}{(2\pi)^{3/2}} \frac{i_s l_s}{i\omega\varepsilon_o\zeta_o} \rho\delta(\rho-\rho_s) \qquad \text{(III.B.185)}$$

where the following Fourier series expansion of the delta function in ϕ has been used:

$$\partial(\phi-\phi_s) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi-\phi_s)} \qquad \text{(III.B.186)}$$

For $\rho \neq \rho_s$ this equation reduces to the expression

$$\left[\rho \frac{\partial}{\partial\rho} \left(\rho \frac{\partial}{\partial\rho}\right) + k_o^2\kappa_t^2\rho^2- \nu^2\right] \Pi_\nu = 0 \qquad \text{(III.B.187)}$$

which, as seen before, has the solution

$$\Pi_\nu = c_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o\kappa_t\rho) \qquad \text{(III.B.188)}$$

For $\rho < \rho_s$ the solution is finite at the origin and therefore

$$\Pi_\nu = c_{\nu<} J_\nu(k_o\kappa_t\rho) \qquad (\rho < \rho_s) \qquad \text{(III.B.189)}$$

For $\rho > \rho_s$ the solution is an outwardly traveling wave and therefore

$$\Pi_\nu = c_{\nu>} H_\nu^{(1)}(k_o\kappa_t\rho) \qquad (\rho > \rho_s) \qquad \text{(III.B.190)}$$

To find the constants $c_{\nu<}$ and $c_{\nu>}$, the boundary condition on Π_ν at $\rho = \rho_s$ must be invoked. The continuity of Π_ν across $\rho = \rho_s$ implies that

$$\Pi_\nu = c_{\nu s} J_\nu(k_o \kappa_t \rho_<)\, H_\nu^{(1)}(k_o \kappa_t \rho_>) \qquad \text{(III.B.191)}$$

where $\rho_<$ is the lesser of ρ and ρ_s and $\rho_>$ is the greater of ρ and ρ_s. The restriction on the derivative of Π_ν is found by substituting the above equation for Π_ν into Eq. III.B.185 and integrating the resulting equation over a small interval containing ρ_s as an interior point. This integration results in the following value for $c_{\nu s}$:

$$c_{\nu s} = -i \frac{1}{4\sqrt{2\pi}} \frac{i_s l_s}{i\omega\varepsilon_o \zeta_o} \qquad \text{(III.B.192)}$$

where use has been made of the Wronskian relation

$$J_\nu(k_o \kappa_t \rho_s)\, H_\nu^{(1)\prime}(k_o \kappa_t \rho_s) - J_\nu'(k_o \kappa_t \rho_s)\, H_\nu^{(1)}(k_o \kappa_t \rho_s) = i \frac{2}{\pi} \frac{1}{k_o \kappa_t \rho_s}$$

(III.B.193)

Therefore

$$\Pi_\nu = -i \frac{1}{4\sqrt{2\pi}} \frac{i_s l_s}{i\omega\varepsilon_o \zeta_o} J_\nu(k_o \kappa_t \rho_<)\, H_\nu^{(1)}(k_o \kappa_t \rho_>) \qquad \text{(III.B.194)}$$

Eqs. III.B.179 are not independent of each other and demonstrate that pure E or H modes no longer exist. Now, however, hybrid modes exist. The terms Π_e or Π_m might now be eliminated between these equations giving a single equation in $(\nabla^t)^2$ and $(\nabla^t)^4$, but it is more convenient to find those linear combinations of Π_e and Π_m that satisfy a first-order equation in $(\nabla^t)^2$. Writing such linear combinations as

$$H_\pm \equiv \alpha_\pm \Pi_e + \beta_\pm \Pi_m \qquad \text{(III.B.195)}$$

and adding $\beta_\pm$ times the second of Eqs. III.B.179 to $\alpha_\pm$ times the first, it is found that these are equations in $H_\pm$ alone of the form

$$[(\nabla^t)^2 + k_o^2 h_\pm^2] H_\pm = 0 \qquad \text{(III.B.196)}$$

provided $\gamma_\pm$ has the values

$$\gamma_\pm \equiv \frac{\beta_\pm}{\alpha_\pm} = k_o^2 \frac{\zeta_e^2 - \zeta_m^2}{2\eta_e} \left(1 \pm \sqrt{1 + \frac{4\hat{\eta}_e \hat{\eta}_m}{(\zeta_e^2 - \zeta_m^2)^2}} \right) \qquad \text{(III.B.197)}$$

where

$$\hat{\eta}_e \hat{\eta}_m \equiv \frac{1}{k_o^4} \eta_e \eta_m \qquad \text{(III.B.198)}$$

The values of $h_\pm$ are then given by

$$h_\pm^2 \equiv \zeta_e^2 - \hat{\gamma}_\pm \eta_e \qquad \text{(III.B.199)}$$

where

$$\hat{\gamma}_\pm \equiv \frac{1}{k_o^2} \gamma_\pm \qquad \text{(III.B.200)}$$

From Eq. III.B.195 the terms Π_e and Π_m must satisfy

$$H_+ = \alpha_+ \Pi_e + \beta_+ \Pi_m \qquad \text{(III.B.201)}$$

$$H_- = \alpha_- \Pi_e + \beta_- \Pi_m \qquad \text{(III.B.202)}$$

so that

$$\Pi_e = \frac{\beta_- H_+ - \beta_+ H_-}{\beta_- \alpha_+ - \beta_+ \alpha_-} \qquad \text{(III.B.203)}$$

$$\Pi_m = \frac{\alpha_- H_+ - \alpha_+ H_-}{\alpha_- \beta_+ - \alpha_+ \beta_-} \qquad \text{(III.B.204)}$$

The problem is over-specified since only the ratios of $\beta_\pm$ to $\alpha_\pm$ have been determined; therefore, two of these constants can be arbitrarily chosen to have any value. For convenience in recovering the isotropic case, α_+ and β_- are chosen to be unity. The isotropic case is then recovered by setting α_- and β_+ to zero. If this choice of constants is made, then

$$H_+ = \Pi_e + \beta_+ \Pi_m = \Pi_e + \gamma_+ \Pi_m \qquad \text{(III.B.205)}$$

$$H_- = \alpha_- \Pi_e + \Pi_m = \Pi_m + \frac{1}{\gamma_-} \Pi_e \qquad \text{(III.B.206)}$$

and

$$\Pi_e = \frac{H_+ - \beta_+ H_-}{1 - \beta_+ \alpha_-} = \frac{H_+ - \gamma_+ H_-}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)} \qquad \text{(III.B.207)}$$

$$\Pi_m = \frac{\alpha_- H_+ - H_-}{\alpha_- \beta_+ - 1} = \frac{H_- - (1/\gamma_-) H_+}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)} \qquad \text{(III.B.208)}$$

where

$$\alpha_- = 1/\gamma_- \qquad \text{(III.B.209)}$$

$$\beta_+ = \gamma_+ \qquad \text{(III.B.210)}$$

Solutions of Eq. III.B.196 may now be sought in a cylindrical coordinate system. In terms of the transformed variables, Eq.

III.B.196 becomes

$$\left[\rho \frac{\partial}{\partial\rho} (\rho \frac{\partial}{\partial\rho}) + k_o^2 h_\pm^2 \rho^2 - \nu^2\right] H_{\nu\pm} = 0 \qquad \text{(III.B.211)}$$

As seen before, these equations are Bessel's equations of the complex arguments $k_o h_\pm \rho$ and integer order ν, i.e.,

$$H_{\nu+} = p_\nu^{(\pm)} Z_\nu^{(\pm)} (k_o h_+ \rho) \qquad \text{(III.B.212)}$$

$$H_{\nu-} = q_\nu^{(\pm)} Z_\nu^{(\pm)} (k_o h_- \rho) \qquad \text{(III.B.213)}$$

Using the above equations for $H_{\nu\pm}$, the equations for $\Pi_{\nu e}$ and $\Pi_{\nu m}$ become

$$\Pi_{\nu e} = \frac{p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o h_+ \rho) - \gamma_+ q_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o h_- \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)} \qquad \text{(III.B.214)}$$

$$\Pi_{\nu m} = \frac{q_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o h_- \rho) - \frac{1}{\gamma_-} p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)} \qquad \text{(III.B.215)}$$

6. Fields

The cylindrical components of the field vectors can now be written in terms of the solutions previously obtained for the vector potentials. In what follows $p_\nu^{(\pm)}$ and $q_\nu^{(\pm)}$ are arbitrary constants.

In a region free of source current in a stationary dielectric, the cylindrical components of the field vectors as derived from Eqs. III.B.104 and III.B.105 are

$$E_\nu^\rho = ik_o^2\gamma\kappa_t\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho) + i\omega\mu_o\eta_o \frac{i\nu}{\rho}\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$E_\nu^\phi = ik_o\gamma\, \frac{i\nu}{\rho}\, p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_o\kappa_t\rho) - i\omega\mu_o\eta_o k_o\kappa_t q_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho)$$

$$E_\nu^z = k_o^2\kappa_t^2\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho) \qquad \text{(III.B.216)}$$

and

$$H_\nu^\rho = ik_o^2\gamma\kappa_t q_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho) - i\omega\varepsilon_o\zeta_o\, \frac{i\nu}{\rho}\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$H_\nu^\phi = ik_o\gamma\, \frac{i\nu}{\rho}\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho) + i\omega\varepsilon_o\zeta_o k_o\kappa_t p_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho)$$

$$H_\nu^z = k_o^2\kappa_t^2\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho) \qquad \text{(III.B.217)}$$

For the source distribution described by Eq. III.B.173 in a stationary dielectric, the x and y components of Π_e and the z components of Π_e and Π_m are nonzero. Therefore, the cylindrical components of the field vectors as derived from Eqs. III.B.110 and III.B.111 in the region $\rho > \rho_s$ are

$$E_\nu^\rho = ik_o^2\gamma\kappa_t\; p_\nu^{(\pm)}\; Z_\nu^{(\pm)\prime}(k_o\kappa_t\rho) + i\omega\mu_o\eta_o\,\frac{i\nu}{\rho}\, q_\nu^{(\pm)}\; Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+\; c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\Big\{\frac{i\nu}{\rho^2}[(t^x - i\nu t^y)\sin\phi - (i\nu t^x + t^y)\cos\phi]H_\nu^{(1)}(k_o\kappa_t\rho)$$

$$\frac{k_o\kappa_t}{\rho}\,[-(i\nu t^x + t^y)\sin\phi + (-t^x + i\nu t^y)\cos\phi]H_\nu^{(1)\prime}(k_o\kappa_t\rho)$$

$$k_o^2\gamma^2(t^y\,\sin\phi + t^x\cos\phi)\; H_\nu^{(1)}(k_o\kappa_t\rho)\Big\}$$

$$E_\nu^\phi = ik_o\gamma\,\frac{i\nu}{\rho}\, p_\nu^{(\pm)}\; Z_\nu^{(\pm)}(k_o\kappa_t\rho) - i\omega\mu_o\eta_o k_o\kappa_t\; q_\nu^{(\pm)}\; Z_\nu^{(\pm)\prime}(k_o\kappa_t\rho)$$

$$+\; c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\Big\{-\frac{i\nu}{\rho^2}(t^y\sin\phi + t^x\cos\phi)H_\nu^{(1)}(k_o\kappa_t\rho)$$

$$\frac{k_o\kappa_t i\nu}{\rho}\,(t^y\sin\phi + t^x\cos\phi)\; H_\nu^{(1)\prime}(k_o\kappa_t\rho)$$

$$k_o^2\kappa_t^2(t^x\sin\phi - t^y\cos\phi)\; H_\nu^{(1)\prime\prime}(k_o\kappa_t\rho)$$

$$k_o^2\gamma^2(-t^x\sin\phi + t^y\cos\phi)\; H_\nu^{(1)}(k_o\kappa_t\rho)\Big\}$$

$$E_\nu^z = k_o^2\kappa_t^2\; p_\nu^{(\pm)}\; Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+\; c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\Big\{ik_o\gamma\,\frac{i\nu}{\rho}(-t^x\sin\phi + t^y\cos\phi)H_\nu^{(1)}(k_o\kappa_t\rho)$$

$$ik_o^2\gamma\kappa_t(t^y\sin\phi + t^x\cos\phi)H_\nu^{(1)\prime}(k_o\kappa_t\rho)\Big\}$$

(III.B.218)

and

$$H_\nu^\rho = ik_o^2\gamma\kappa_t\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)\prime}(k_o\kappa_t\rho) - i\omega\varepsilon_o\zeta_o\,\frac{i\nu}{\rho}\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+ c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\left\{-i\omega\varepsilon_o\zeta_o\, ik_o\gamma(t^x\sin\phi - t^y\cos\phi)H_\nu^{(1)}(k_o\kappa_t\rho)\right\}$$

$$H_\nu^\phi = ik_o\gamma\,\frac{i\nu}{\rho}\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho) + i\omega\varepsilon_o\zeta_o k_o\kappa_t\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)\prime}(k_o\kappa_t\rho)$$

$$+ c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\left\{-i\omega\varepsilon_o\zeta_o ik_o\gamma(t^y\sin\phi + t^x\cos\phi)H_\nu^{(1)}(k_o\kappa_t\rho)\right\}$$

$$H_\nu^z = k_o^2\kappa_t^2\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+ c_{\nu s}J_\nu(k_o\kappa_t\rho_s)\times\left\{i\omega\varepsilon_o\zeta_o\,\frac{i\nu}{\rho}(t^y\sin\phi + t^x\cos\phi)H_\nu^{(1)}(k_o\kappa_t\rho)\right.$$

$$\left. - i\omega\varepsilon_o\zeta_o k_o\kappa_t(-t^x\sin\phi + t^y\cos\phi)H_\nu^{(1)\prime}(k_o\kappa_t\rho)\right\}$$

(III.B.219)

When the trigonometric terms in Eqs. III.B.218 and III.B.219 are replaced with their equivalent exponential representations, and after the terms of order ν only are regrouped, the cylindrical components of the field vectors become

$$E_\nu^\rho = ik_o^2\gamma\kappa_t\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho) + i\omega\mu_o\eta_o \frac{i\nu}{\rho}\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+ \frac{k_o\kappa_t}{\rho}\frac{c_{\nu s}}{2}\{[(t^x - i\nu t^y) - i(i\nu t^x + t^y)]\, e^{i\phi_s}\, J_{\nu-1}(k_o\kappa_t\rho_s)$$

$$+ [-(t^x - i\nu t^y) - i(i\nu t^x + t^y)]\, e^{-i\phi_s}\, J_{\nu+1}(k_o\kappa_t\rho_s)\}\, H_\nu^{(1)}(k_o\kappa_t\rho)$$

$$+ k_o^2\gamma^2 \frac{c_{\nu s}}{2}[(t^x - it^y)e^{i\phi_s}\, J_{\nu-1}(k_o\kappa_t\rho_s)H_{\nu-1}^{(1)}(k_o\kappa_t\rho)$$

$$- (-t^x - it^y)\, e^{-i\phi_s}\, J_{\nu+1}(k_o\kappa_t\rho_s)H_{\nu+1}^{(1)}(k_o\kappa_t\rho)]$$

$$E_\nu^\phi = ik_o\gamma \frac{i\nu}{\rho}\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho) - i\omega\mu_o\eta_o k_o\kappa_t\, q_\nu^{(\pm)}\, Z_\nu^{(\pm)'}(k_o\kappa_t\rho)$$

$$+ k_o^2\kappa_t^2 \frac{c_{\nu s}}{2}[(it^x + t^y)e^{i\phi_s}\, J_{\nu-1}(k_o\kappa_t\rho_s)$$

$$+ (it^x - t^y)\, e^{-i\phi_s}\, J_{\nu+1}(k_o\kappa_t\rho_s)]\, H_\nu^{(1)'}(k_o\kappa_t\rho)$$

$$+ k_o^2\gamma^2 \frac{c_{\nu s}}{2}[(it^x + t^y)e^{i\phi_s}\, J_{\nu-1}(k_o\kappa_t\rho_s)\, H_{\nu-1}^{(1)}(k_o\kappa_t\rho)$$

$$- (it^x - t^y)\, e^{-i\phi_s}\, J_{\nu+1}(k_o\kappa_t\rho_s)H_{\nu+1}^{(1)}(k_o\kappa_t\rho)]$$

$$E_\nu^z = k_o^2\kappa_t^2\, p_\nu^{(\pm)}\, Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+ k_o^2\gamma\kappa_t \frac{c_{\nu s}}{2}[-(it^x + t^y)e^{i\phi_s}\, J_{\nu-1}(k_o\kappa_t\rho_s)$$

$$+ (it^x - t^y)e^{-i\phi_s}\, J_{\nu+1}(k_o\kappa_t\rho_s)]\, H_\nu^{(1)}(k_o\kappa_t\rho) \qquad \text{(III.B.220)}$$

and

$$H_\nu^\rho = ik_o^2\gamma\kappa_t\ q_\nu^{(\pm)}\ Z_\nu^{(\pm)'}(k_o\kappa_t\rho) - i\omega\varepsilon_o\zeta_o \frac{i\nu}{\rho} p_\nu^{(\pm)}\ Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$+ i\omega\varepsilon_o\zeta_o ik_o\gamma \frac{c_{\nu s}}{2} [(it^x + t^y)e^{i\phi_s} J_{\nu-1}(k_o\kappa_t\rho_s)H_{\nu-1}^{(1)}(k_o\kappa_t\rho)$$

$$- (it^x - t^y)e^{-i\phi_s} J_{\nu+1}(k_o\kappa_t\rho_s)\ H_{\nu+1}^{(1)}(k_o\kappa_t\rho)]$$

$$H_\nu^\phi = ik_o\gamma \frac{i\nu}{\rho} q_\nu^{(\pm)}\ Z_\nu^{(\pm)}(k_o\kappa_t\rho) + i\omega\varepsilon_o\zeta_o k_o\kappa_t\ p_\nu^{(\pm)}\ Z_\nu^{(\pm)'}(k_o\kappa_t\rho)$$

$$- i\omega\varepsilon_o\zeta_o\ ik_o\gamma \frac{c_{\nu s}}{2} [(t^x - it^y)e^{i\phi_s} J_{\nu-1}(k_o\kappa_t\rho_s)H_{\nu-1}^{(1)}(k_o\kappa_t\rho)$$

$$+ (t^x + it^y)\ e^{-i\phi_s} J_{\nu+1}(k_o\kappa_t\rho_s)\ H_{\nu+1}^{(1)}(k_o\kappa_t\rho)]$$

$$H_\nu^z = k_o^2\kappa_t^2\ q_\nu^{(\pm)}\ Z_\nu^{(\pm)}(k_o\kappa_t\rho)$$

$$i\omega\varepsilon_o\zeta_o ik_o\kappa_t \frac{c_{\nu s}}{2} [(t^x - it^y)e^{i\phi_s} J_{\nu-1}(k_o\kappa_t\rho_s)$$

$$+ (t^x + it^y)\ e^{-i\phi_s} J_{\nu+1}(k_o\kappa_t\rho_s)]H_\nu^{(1)}(k_o\kappa_t\rho) \qquad \text{(III.B.221)}$$

In arriving at these results, the following relations were used:

$$H_\nu^{(1)}(k_o\kappa_t\rho) = \frac{\nu \pm 1}{k_o\kappa_t\rho} H_{\nu\pm1}^{(1)}(k_o\kappa_t\rho) \pm H_{\nu\pm1}^{(1)'}(k_o\kappa_t\rho) \qquad \text{(III.B.222)}$$

$$H_\nu^{(1)'}(k_o\kappa_t\rho) = -\frac{\nu \pm 1}{k_o^2\kappa_t^2\rho^2} H_{\nu\pm1}^{(1)}(k_o\kappa_t\rho) + \frac{\nu \pm 1}{k_o\kappa_t\rho} H_{\nu\pm1}^{(1)'}(k_o\kappa_t\rho)$$

$$\pm H_{\nu\pm1}^{(1)''}(k_o\kappa_t\rho) \qquad \text{(III.B.223)}$$

These relations can be derived from the recurrence relations (14)

$$H^{(1)}_{\nu-1}(k_o\kappa_t\rho) + H^{(1)}_{\nu+1}(k_o\kappa_t\rho) = \frac{2\nu}{k_o\kappa_t\rho} H^{(1)}_{\nu}(k_o\kappa_t\rho) \qquad \text{(III.B.224)}$$

$$H^{(1)}_{\nu-1}(k_o\kappa_t\rho) - H^{(1)}_{\nu+1}(k_o\kappa_t\rho) = 2H^{(1)'}_{\nu}(k_o\kappa_t\rho) \qquad \text{(III.B.225)}$$

In a region free of source currents in a stationary plasma, the cylindrical components of the field vectors as derived from Eqs. III.B.171 and III.B.172 are

$$E^{\rho}_{\nu} = ik_o^2\gamma \, \frac{h_+\rho^{(\pm)}_{\nu} \, Z^{(\pm)'}_{\nu}(k_o h_+\rho) - h_-\gamma_+ q^{(\pm)}_{\nu} \, Z^{(\pm)'}_{\nu}(k_o h_-\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- \frac{k_o\gamma\zeta_+}{\zeta_\perp - \gamma^2} \, \frac{i\nu}{\rho} \, \frac{\rho^{(\pm)}_{\nu} \, Z^{(\pm)}_{\nu}(k_o h_+\rho) - \gamma_+ \, q^{(\pm)}_{\nu} \, Z^{(\pm)}_{\nu}(k_o h_-\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ \frac{\omega\mu_o\zeta_+}{\zeta_\perp - \gamma^2} \, k_o \, \frac{h_- q^{(\pm)}_{\nu} \, Z^{(\pm)'}_{\nu}(k_o h_-\rho) - h_+ \frac{1}{\gamma_-} p^{(\pm)}_{\nu} \, Z^{(\pm)'}_{\nu}(k_o h_+\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ i\omega\mu_o \, \frac{i\nu}{\rho} \, \frac{q^{(\pm)}_{\nu} \, Z^{(\pm)}_{\nu}(k_o h_-\rho) - \frac{1}{\gamma_-} p^{(\pm)}_{\nu} \, Z^{(\pm)}_{\nu}(k_o h_+\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$E_\nu^\phi = ik_o\gamma \frac{i\nu}{\rho} \frac{p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_+\rho) - \gamma_+ q_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_-\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ \frac{k_o\gamma\zeta_+}{\zeta_\perp - \gamma^2} k_o \frac{h_+p_\nu^{(\pm)} Z_\nu^{(\pm)'}(k_oh_+\rho) - h_-\gamma_+q_\nu^{(\pm)} Z_\nu^{(\pm)'}(k_oh_-\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ \frac{\omega\mu_o\zeta_+}{\zeta_\perp - \gamma^2} \frac{i\nu}{\rho} \frac{q_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_-\rho) - \frac{1}{\gamma_-} p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_+\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- i\omega\mu_o k_o \frac{h_-q_\nu^{(\pm)} Z_\nu^{(\pm)'}(k_oh_-\rho) - h_+ \frac{1}{\gamma_-} p_\nu^{(\pm)} Z_\nu^{(\pm)'}(k_oh_+\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$E_\nu^z = k_o^2 \frac{\Delta}{\zeta_\perp - \gamma^2} \frac{p_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_+\rho) - \gamma_+q_\nu^{(\pm)} Z_\nu^{(\pm)}(k_oh_-\rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)} \qquad \text{(III.B.226)}$$

and

$$H_\nu^\rho = ik_o^2\gamma \; \frac{h_- q_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_- \rho) - h_+ \frac{1}{\gamma_-} p_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- \frac{k_o \gamma \zeta_+}{\zeta_\perp - \gamma^2} \frac{i\nu}{\rho} \; \frac{q_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_- \rho) - \frac{1}{\gamma_-} p_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- \frac{\omega \varepsilon_o \zeta_+ \gamma^2}{\zeta_\perp - \gamma^2} k_o \; \frac{h_+ p_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_+ \rho) - h_- \gamma_+ q_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_- \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- i\omega\varepsilon_o (\zeta_\perp - \frac{\zeta_+^2}{\zeta_\perp - \gamma^2}) \frac{i\nu}{\rho} \; \frac{p_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_+ \rho) - \gamma_+ \; q_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_- \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$H_\nu^\phi = ik_o \gamma \frac{i\nu}{\rho} \; \frac{q_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_- \rho) - \frac{1}{\gamma_-} p_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ \frac{k_o \gamma \zeta_+}{\zeta_\perp - \gamma^2} k_o \; \frac{h_- q_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_- \rho) - h_+ \frac{1}{\gamma_-} p_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$- \frac{\omega \varepsilon_o \zeta_+ \; \gamma^2}{\zeta_\perp - \gamma^2} \frac{i\nu}{\rho} \; \frac{p_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_+ \rho) - \gamma_+ q_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_- \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$+ i\omega\varepsilon_o (\zeta_\perp - \frac{\zeta_+^2}{\zeta_\perp - \gamma^2}) k_o \; \frac{h_+ p_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_+ \rho) - h_- \gamma_+ q_\nu^{(\pm)} \; Z_\nu^{(\pm)'}(k_o h_- \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

$$H_\nu^z = k_o^2 \; \frac{\Delta}{\zeta_\perp - \gamma^2} \; \frac{q_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_- \rho) - \frac{1}{\gamma_-} p_\nu^{(\pm)} \; Z_\nu^{(\pm)}(k_o h_+ \rho)}{1 - (\hat{\gamma}_+/\hat{\gamma}_-)}$$

(III.B.227)

C. Moving Media

1. Introduction

Now that the field vectors in a stationary medium have been determined, the problem remains of determining how the field vectors behave in the presence of a moving medium. This is accomplished by studying certain aspects of the Special Theory of Relativity (15). Since the experimental basis and the development of the theory are described in detail in many places, only a brief summary of the key points needed in this study will be presented.

The following notation will be used throughout the remainder of this study. The time coordinate t is denoted by x^o and the space coordinate $\underline{r}$ is separated into the rectangular components x^1, x^2, and x^3. If the time coordinate is measured in the same units as the space coordinates, then the mathematical expressions presented in the remainder of this study have a more symmetrical form. This is accomplished by arbitrarily setting the speed of light in vacuum equal to unity. This set of geometrized units is used throughout the remainder of the study. An event is then defined as a point in space-time and is denoted by $\underset{\sim}{x} \equiv (x^o, x^1, x^2, x^3)$ or x^μ . Actually, the term x^μ denotes the contravariant components of a point in space-time. The covariant components x_μ can be formed by lowering the contravariant components with the metric tensor, i.e.,*

---

*In this study the Einstein summation convention is used. That is, an upper and lower repeated index are understood to be summed on, even though the summation sign is not written. If the repeated index is Roman, the sum is from 1-3; if the index is Greek, the sum is from 0-3.

$$x^{\mu} = \eta^{\mu\nu} x_{\nu} \qquad \text{(III.C.1)}$$

or

$$x_{\mu} = \eta_{\mu\nu} x^{\nu} \qquad \text{(III.C.2)}$$

The metric of the flat spacetime of special relativity theory is

$$\| \eta^{\mu\nu} \| \equiv \| \eta_{\mu\nu} \| \equiv \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \qquad \text{(III.C.3)}$$

2. The Lorentz Transformation

The Lorentz transformation relates the coordinates x^{μ} in the inertial reference frame s and the coordinates $x^{\mu'}$ in the inertial reference frame s' . The Lorentz transformation can be considered a consequence of the postulate that the speed of light in vacuum has the same value in all inertial reference frames. To derive the Lorentz transformation it is only necessary to assume that the transformation is linear. This seems very plausible and is equivalent to the assumption that space-time is homogeneous and isotropic. With this assumption, the Lorentz transformation of the coordinates between the inertial rest frames s and s' becomes

$$x^{\mu} = \Lambda^{\mu}{}_{\nu'} x^{\nu'} \qquad \text{(III.C.4)}$$

or

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \qquad \text{(III.C.5)}$$

where

$$||\Lambda^{\mu}{}_{\nu'}|| \equiv \left[\begin{array}{c|c} \gamma & \beta\gamma\hat{\beta} \\ \hline \beta\gamma\hat{\beta} & \underline{\underline{u}} + \hat{\beta}\hat{\beta}(\gamma - 1) \end{array}\right] \tag{III.C.6}$$

$$||\Lambda^{\mu'}{}_{\nu}|| \equiv \left[\begin{array}{c|c} \gamma & -\beta\gamma\hat{\beta} \\ \hline -\beta\gamma\hat{\beta} & \underline{\underline{u}} + \hat{\beta}\hat{\beta}(\gamma - 1) \end{array}\right] \tag{III.C.7}$$

In Eqs. III.C.6 and III.C.7, β represents the velocity of frame s' relative to frame s, and

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \tag{III.C.8}$$

In the previous notation Eq. III.C.4 implies that

$$\begin{aligned} t &= \gamma(t' + \beta\hat{\beta} \cdot \underline{r}') \\ \underline{r} &= [\underline{\underline{u}} + (\gamma - 1)\hat{\beta}\hat{\beta}] \cdot \underline{r}' + \beta\gamma\hat{\beta}\, t' \end{aligned} \tag{III.C.9}$$

If one assumes that the coordinates undergo a proper Lorentz transformation, then a 4-vector is defined as a set of four quantities v^{μ} that transform like the coordinates

$$v^{\mu} = \Lambda^{\mu}{}_{\nu'}\, v^{\nu'} \tag{III.C.10}$$

or

$$v^{\mu'} = \Lambda^{\mu'}{}_{\nu'}\, v^{\nu'} \tag{III.C.11}$$

and a 4-dyad is defined as a set of 4^2 quantities $d^{\mu\nu}$ that obey the transformation law

$$d^{\mu\nu} = \Lambda^{\mu}{}_{\zeta'} \Lambda^{\nu}{}_{\eta'} d^{\zeta'\eta'} \qquad (\text{III.C.12})$$

or

$$d^{\mu'\nu'} = \Lambda^{\mu'}{}_{\zeta} \Lambda^{\nu'}{}_{\eta} d^{\zeta\eta} \qquad (\text{III.C.13})$$

3. The Covariance of Maxwell's Equations

The electric field vector $\underline{E}$ and the magnetic field vector $\underline{B}$ can be written as the elements of an antisymmetric field dyad $\mathfrak{f}^{\mu\nu}$ such that the two homogeneous Maxwell equations

$$\begin{aligned} \underline{\nabla} \wedge \underline{E} + \frac{\partial \underline{B}}{\partial t} &= 0 \\ \underline{\nabla} \cdot \underline{B} &= 0 \end{aligned} \qquad (\text{III.C.14})$$

can be written in the covariant form

$$\frac{\partial}{\partial x^{\nu}} \, {*\mathfrak{f}}^{\mu\nu} = 0 \qquad (\text{III.C.15})$$

The dual operation $*(\;)$ is defined as

$$*d^{\mu\nu} \equiv \frac{1}{2!} \varepsilon^{\mu\nu\zeta\eta} d_{\zeta\eta} \qquad (\text{III.C.16})$$

for any antisymmetric dyad where

$$\varepsilon^{\alpha\beta\gamma\delta} \equiv \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ form an even permutation of } 0123 \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ form an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases} \qquad (\text{III.C.17})$$

Explicitly, the field dyad is

$$\|f^{\mu\nu}\| \equiv \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{bmatrix} \tag{III.C.18}$$

Similarly, the electric field vector $\underline{D}$ and the magnetic field vector $\underline{H}$ can be written as the elements of an antisymmetric field dyad $g^{\mu\nu}$ such that the two inhomogeneous Maxwell equations

$$\begin{aligned} \underline{\nabla} \cdot \underline{D} &= \rho_o \\ \underline{\nabla} \wedge \underline{H} - \frac{\partial \underline{D}}{\partial t} &= \underline{j}_o \end{aligned} \tag{III.C.19}$$

can be written in the covariant form

$$\frac{\partial}{\partial x^\nu} g^{\mu\nu} = j_o^\mu \tag{III.C.20}$$

The source charge-density 4-vector j_o^μ is defined as

$$\underset{\sim}{j}_o \equiv (\rho_o, \underline{j}_o) \tag{III.C.21}$$

Explicitly, the field dyad is

$$\|g^{\mu\nu}\| \equiv \begin{bmatrix} 0 & D^x & D^y & D^z \\ -D^x & 0 & H^z & -H^y \\ -D^y & -H^z & 0 & H^x \\ -D^z & H^y & -H^x & 0 \end{bmatrix} \tag{III.C.22}$$

4. Transformations

Since the fields $\underline{E}$ and $\underline{B}$ are elements of the field dyad $f^{\mu\nu}$, their transformation properties can be found from

$$f^{\mu\nu} = \Lambda^{\mu}{}_{\zeta'} \Lambda^{\nu}{}_{\eta'} f^{\zeta'\eta'} \tag{III.C.23}$$

With the transformation III.C.6 from the system s' to s , the above equation gives the transformed fields*

$$\begin{aligned} \underline{E} &= \{\gamma \underline{\underline{u}} + (1-\gamma)\underline{\hat{\beta}}\,\underline{\hat{\beta}}\} \cdot \underline{E}' - \beta\gamma\underline{\underline{c}} \cdot \underline{B}' \\ \underline{B} &= \{\gamma \underline{\underline{u}} + (1-\gamma)\underline{\hat{\beta}}\,\underline{\hat{\beta}}\} \cdot \underline{B}' + \beta\gamma\underline{\underline{c}} \cdot \underline{E}' \end{aligned} \tag{III.C.24}$$

Similarly, since the fields $\underline{D}$ and $\underline{H}$ are elements of the field dyad $g^{\mu\nu}$, their transformation properties can be found from

$$g^{\mu\nu} = \Lambda^{\mu}{}_{\zeta'} \Lambda^{\nu}{}_{\eta'}\, g^{\zeta'\eta'}$$

With the transformation III.C.6 from the system s' to s the above equation gives the transformed fields*

$$\begin{aligned} \underline{D} &= \{\gamma \underline{\underline{u}} + (1-\gamma)\, \underline{\hat{\beta}}\,\underline{\hat{\beta}}\} \cdot \underline{D}' - \beta\gamma\, \underline{\underline{c}} \cdot \underline{H}' \\ \underline{H} &= \{\gamma\underline{\underline{u}} + (1-\gamma)\, \underline{\hat{\beta}}\,\underline{\hat{\beta}}\} \cdot \underline{H}' + \beta\gamma\, \underline{\underline{c}} \cdot \underline{D}' \end{aligned} \tag{III.C.26}$$

The phase ϕ of a wave in the system s is defined by

$$\phi \equiv \underset{\sim}{k} \cdot \underset{\sim}{r} \equiv k_o \underline{\kappa} \cdot \underline{r} - \omega t \tag{III.C.27}$$

---

*The dyad $\underline{\underline{c}}$ is now referenced to the direction of $\hat{\beta}$

Since the frequency ω and the wave 3-vector $k_o \underline{\kappa}$ are the elements of a wave 4-vector $\underset{\sim}{k}$, their properties can be found from

$$k^{\mu} = \Lambda^{\mu}_{\ \nu'} \, k^{\nu'} \tag{III.C.28}$$

With the transform III.C.6 from the system s' to s, the above equation gives the transformed variables

$$\begin{aligned} \omega &= \gamma(\omega' + k_o' \, \underline{\kappa}' \cdot \hat{\beta} \, \beta) \\ k_o \underline{\kappa} &= [\underline{\underline{u}} + (\gamma-1)\hat{\beta} \, \hat{\beta}] \cdot \underline{\kappa}' k_o' + \beta \gamma \hat{\beta} \omega' \end{aligned} \tag{III.C.29}$$

From the invariance of the phase of a wave

$$k_o \underline{\kappa} \cdot \underline{r} - \omega t = k_o' \underline{\kappa}' \cdot \underline{r}' - \omega' t' \tag{III.C.30}$$

These results also hold in terms of the transformed variables ω and $k_o \gamma$. For the special case of relative motion in the z direction, i.e.,

$$\underline{\beta} \equiv \hat{z} \, \beta_z \qquad \text{or} \qquad (\hat{\beta} \equiv \hat{z}) \tag{III.C.31}$$

and

$$\gamma_z \equiv \frac{1}{\sqrt{1-\beta_z^2}} \tag{III.C.32}$$

Equations III.C.9 for the transformed coordinates reduce to the expressions

$$\begin{aligned} t &= \gamma_z(t' + \beta_z z') \\ z &= \gamma_z(z' + \beta_z t') \\ \underline{r}^t &= \underline{r}^{t'} \end{aligned} \tag{III.C.33}$$

Equations III.C.24 and III.C.26 for the transformed fields reduce to the expressions

$$
\begin{aligned}
E^z &= E^{z'} & \underline{E}^t &= \gamma_z(\underline{E}^{t'} - \beta_z \underline{\underline{c}} \cdot \underline{B}^{t'}) \\
B^z &= B^{z'} & \underline{B}^t &= \gamma_z(\underline{B}^{t'} + \beta_z \underline{\underline{c}} \cdot \underline{E}^{t'}) \\
D^z &= D^{z'} & \underline{D}^t &= \gamma_z(\underline{D}^{t'} - \beta_z \underline{\underline{c}} \cdot \underline{H}^{t'}) \\
H^z &= H^{z'} & \underline{H}^t &= \gamma_z(\underline{H}^{t'} + \beta_z \underline{\underline{c}} \cdot \underline{D}^{t'})
\end{aligned}
\qquad \text{(III.C.34)}
$$

Similarly, Eq. III.C.29 for ω and $k_o\underline{\kappa}$ reduce to the expressions

$$
\begin{aligned}
\omega &= \gamma_z(\omega' + \beta_z k_o' \gamma') \\
k_o \gamma &= \gamma_z(k_o'\gamma' + \beta_z \omega') \\
k_o \underline{\kappa}^t &= k_o' \underline{\kappa}^{t'}
\end{aligned}
\qquad \text{(III.C.35)}
$$

also

$$
k_o \gamma z - \omega t = k_o' \gamma' z' - \omega' t' \qquad \text{(III.C.36)}
$$

5. Constitutive Relations

In the system s' the constitutive relations in an anisotropic plasma are

$$
\underline{D}' = \epsilon_o \underline{\underline{\zeta}}' \cdot \underline{E}' = \epsilon_o[(\zeta_\perp' \underline{\underline{t}} + i\zeta_+' \underline{\underline{c}}) \cdot \underline{E}^{t'} + \hat{z}\zeta_\parallel' E^{z'}] \qquad \text{(III.C.37)}
$$

$$
\underline{H}' = \frac{1}{\mu_o} (\underline{\underline{n}}^{-1})' \cdot \underline{B}' = \frac{1}{\mu_o} \underline{B}' \qquad \text{(III.C.38)}
$$

When the primed field variables in s' are expressed in terms of the unprimed field variables in s with the aid of Eqs. III.C.34, the constitutive relations become

$$\gamma_z(\underline{D}^t + \beta_z \underline{\underline{c}} \cdot \underline{H}^t) + \hat{z}\, D^z = \varepsilon_o(\zeta'_\perp \underline{\underline{t}} + i\zeta'_+ \underline{\underline{c}}) \cdot \gamma_z(\underline{E}^t + \beta_z \underline{\underline{c}} \cdot \underline{B}^t) + \hat{z}\, \varepsilon_o \zeta'_\parallel E^z \qquad \text{(III.C.39)}$$

$$\gamma_z(\underline{B}^t - \beta_z \underline{\underline{c}} \cdot \underline{E}^t) + \hat{z}\, B^z = \mu_o \gamma_z(\underline{H}^t - \beta_z \underline{\underline{c}} \cdot \underline{D}^t) + \hat{z}\, \mu_o H^z \qquad \text{(III.C.40)}$$

The axial and transverse components of these equations can be separated, as follows:

$$\left.\begin{aligned} D^z &= \varepsilon_o \zeta'_\parallel E^z \\ H^z &= \frac{1}{\mu_o} B^z \end{aligned}\right\} \text{axial} \qquad \text{(III.C.41)}$$

$$\left.\begin{aligned} \underline{D}^t + \beta_z \underline{\underline{c}} \cdot \underline{H}^t &= \varepsilon_o(\zeta'_\perp \underline{\underline{t}} + i\zeta'_+ \underline{\underline{c}}) \cdot (\underline{E}^t + \beta_z \underline{\underline{c}} \cdot \underline{B}^t) \\ \underline{B}^t - \beta_z \underline{\underline{c}} \cdot \underline{E}^t &= \mu_o(\underline{H}^t - \beta_z \underline{\underline{c}} \cdot \underline{D}^t) \end{aligned}\right\} \text{transverse} \qquad \text{(III.C.42)}$$

After some rearrangement the transverse equations become

$$\underline{\underline{t}} \cdot \underline{D}^t + \beta_z \underline{\underline{c}} \cdot \underline{H}^t = \varepsilon_o(\zeta'_\perp \underline{\underline{t}} + i\zeta'_+ \underline{\underline{c}}) \cdot \underline{E}^t + \varepsilon_o \beta_z(\zeta'_\perp \underline{\underline{c}} - i\zeta'_+ \underline{\underline{t}}) \cdot \underline{B}^t \qquad \text{(III.C.43)}$$

$$-\beta_z \underline{\underline{c}} \cdot \underline{D}^t + \underline{\underline{t}} \cdot \underline{H}^t = -\frac{1}{\mu_o} \beta_z \underline{\underline{c}} \cdot \underline{E}^t + \frac{1}{\mu_o} \underline{\underline{t}} \cdot \underline{B}^t \qquad \text{(III.C.44)}$$

It is possible to solve Eqs. III.C.43 and III.C.44 simultaneously for the transverse field vectors $\underline{D}^t$ and $\underline{H}^t$ in terms of the transverse field vectors $\underline{E}^t$ and $\underline{B}^t$. The constitutive relations in the system s then become

$$\underline{D}^t = \gamma_z^2\beta_z \left[\left(\frac{1}{\beta_z}\,\varepsilon_o\zeta'_\perp - \beta_z\,\frac{1}{\mu_o}\right)\underline{\underline{t}} + \frac{1}{\beta_z}\,\varepsilon_o i\zeta'_+\,\underline{\underline{c}}\right]\cdot\underline{E}^t$$

$$+ \gamma_z^2\beta_z\left[\left(\varepsilon_o\zeta'_\perp - \frac{1}{\mu_o}\right)\underline{\underline{c}} - \varepsilon_o i\zeta'_+\underline{\underline{t}}\right]\cdot\underline{B}^t \qquad \text{(III.C.45)}$$

$$\underline{H}^t = \gamma_z^2\beta_z \left[\left(\frac{1}{\beta_z}\,\frac{1}{\mu_o} - \beta_z\varepsilon_o\zeta'_\perp\right)\underline{\underline{t}} - \beta_z\varepsilon_o i\zeta'_+\underline{\underline{c}}\right]\cdot\underline{B}^t$$

$$+ \gamma_z^2\beta_z\left[\left(\varepsilon_o\zeta'_\perp - \frac{1}{\mu_o}\right)\underline{\underline{c}} - \varepsilon_o i\zeta'_+\underline{\underline{t}}\right]\cdot\underline{E}^t \qquad \text{(III.C.46)}$$

D. Fields in the Rest Frame of the Antenna

1. Introduction

Using the results derived in the previous section on the special theory of relativity, one can now determine the fields in the presence of the moving plasma flow field.

The inertial reference frame s is taken to be at rest with respect to the antenna. In the i^{th} cylindrical layer of the plasma $(1 \leq i \leq n)$, an inertial reference frame s_i' is taken to be at rest with respect to the plasma in that layer. Note that the velocities in adjacent layers need not be the same.

2. Region 0

In the region 0 , let

$$\zeta_o \equiv \varepsilon_{ro} + i \frac{\sigma_o}{\omega \varepsilon_o} \tag{III.D.1}$$

$$\eta_o \equiv \mu_{ro} \tag{III.D.2}$$

$$\kappa_o^2 \equiv \zeta_o \eta_o \tag{III.D.3}$$

$$\kappa_{to}^2 \equiv \kappa_o^2 - \gamma^2 \tag{III.D.4}$$

where ε_{ro} is the relative permittivity, μ_{ro} is the relative permeability, and σ_o is the conductivity of the region 0 .

Since the region 0 contains the origin, the linear combination $p_{\nu o}^{(\pm)} Z_\nu^{(\pm)}(k_o \kappa_{to} \rho)$ becomes

$$p_{\nu o} J_\nu(k_o \kappa_{to} \rho) \tag{III.D.5}$$

and the linear combination $q_{\nu o}^{(\pm)} Z_{\nu}^{(\pm)}(k_o \kappa_{to} \rho)$ becomes

$$q_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) \tag{III.D.6}$$

where $p_{\nu o}$ and $q_{\nu o}$ are arbitrary constants to be determined later in this study.

In the rest frame of the antenna in the region 0, the field vectors as derived from Eqs. III.B.220 and III.B.221 are

$$\begin{aligned}
E_{\nu o}^{\rho} &= ik_o^2 \gamma \kappa_{to} p_{\nu o} J_{\nu}'(k_o \kappa_{to} \rho) + i\omega\mu_o \eta_o \frac{i\nu}{\rho} q_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) + e_{\nu o}^{\rho} \\
E_{\nu o}^{\emptyset} &= ik_o \gamma \frac{i\nu}{\rho} p_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) - i\omega\mu_o \eta_o k_o \kappa_{to} q_{\nu o} J_{\nu}'(k_o \kappa_{to} \rho) + e_{\nu o}^{\emptyset} \\
E_{\nu o}^{z} &= k_o^2 \kappa_{to}^2 p_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) + e_{\nu o}^{z}
\end{aligned} \tag{III.D.7}$$

and

$$\begin{aligned}
H_{\nu o}^{\rho} &= ik_o^2 \gamma \kappa_{to} q_{\nu o} J_{\nu}'(k_o \kappa_{to} \rho) - i\omega\varepsilon_o \zeta_o \frac{i\nu}{\rho} p_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) + h_{\nu o}^{\rho} \\
H_{\nu o}^{\emptyset} &= ik_o \gamma \frac{i\nu}{\rho} q_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) + i\omega\varepsilon_o \zeta_o k_o \kappa_{to} p_{\nu o} J_{\nu}'(k_o \kappa_{to} \rho) + h_{\nu o}^{\emptyset} \\
H_{\nu o}^{z} &= k_o^2 \kappa_{to}^2 q_{\nu o} J_{\nu}(k_o \kappa_{to} \rho) + h_{\nu o}^{z}
\end{aligned} \tag{III.D.8}$$

where

$$e^{\rho}_{\nu o} \equiv + \frac{k_o \kappa_{to}}{\rho} \frac{c_{\nu s}}{2} \{ [(t^x - i\nu t^y) - i(i\nu t^x + t^y)] \, e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s)$$

$$+ [-(t^x - i\nu t^y) - i(i\nu t^x + t^y)] e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s) \} \, H^{(1)}_{\nu}(k_o \kappa_{to} \rho)$$

$$+ k_o^2 \gamma^2 \frac{c_{\nu s}}{2} [(t^x - it^y) \, e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu-1}(k_o \kappa_{to} \rho)$$

$$- (-t^x - it^y) \, e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu+1}(k_o \kappa_{to} \rho)]$$

$$e^{\phi}_{\nu o} \equiv + k_o^2 \kappa_{to}^2 \frac{c_{\nu s}}{2} [(it^x + t^y) \, e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s) + (it^x - t^y) e^{-i\phi_s} \, J_{\nu+1}$$

$$\times (k_o \kappa_{to} \rho_s)] \, H^{(1)\prime}_{\nu}(k_o \kappa_{to} \rho) + k_o^2 \gamma^2 \frac{c_{\nu s}}{2} [(it^x + t^y) e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s)$$

$$\times \; H^{(1)}_{\nu-1}(k_o \kappa_{to} \rho) - (it^x - t^y) \, e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu+1}(k_o \kappa_{to} \rho)]$$

$$e^{z}_{\nu o} \equiv + k_o^2 \gamma \kappa_{to} \frac{c_{\nu s}}{2} [-(it^x + t^y) e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s)$$

$$+ (it^x - t^y) \, e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s)] \, H^{(1)}_{\nu}(k_o \kappa_{to} \rho) \qquad \text{(III.D.7)}$$

$$h^{\rho}_{\nu o} \equiv + i\omega \varepsilon_o \zeta_o i k_o \gamma \frac{c_{\nu s}}{2} [(it^x + t^y) \, e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu-1}(k_o \kappa_{to} \rho)$$

$$-(it^x - t^y) e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu+1}(k_o \kappa_{to} \rho)]$$

$$h^{\phi}_{\nu o} \equiv \; i\omega \varepsilon_o \zeta_o i k_o \, \gamma \; \frac{c_{\nu s}}{2} [(t^x - it^y) e^{i\phi_s} \, J_{\nu-1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu-1}(k_o \kappa_{to} \rho)$$

$$+ (t^x + it^y) e^{-i\phi_s} \, J_{\nu+1}(k_o \kappa_{to} \rho_s) H^{(1)}_{\nu+1}(k_o \kappa_{to} \rho)]$$

$$h^{z}_{\nu o} \equiv i\omega\varepsilon_o\zeta_o \; ik_o\kappa_{to} \frac{c_{\nu s}}{2}[(t^x - it^y)e^{i\phi_s} J_{\nu-1}(k_o\kappa_{to}\rho_s)$$

$$+ (t^x + it^y)e^{-i\phi_s} J_{\nu+1}(k_o\kappa_{to}\rho_s)] \; H^{(1)}_{\nu}(k_o\kappa_{to}\rho)$$

3. Region ∞

In the region ∞ , let

$$\zeta_\infty \equiv \varepsilon_{r\infty} + i \frac{\sigma_\infty}{\omega\varepsilon_o} \tag{III.D.10}$$

$$\eta_\infty \equiv \mu_{r\infty} \tag{III.D.11}$$

$$\kappa^2_\infty \equiv \zeta_\infty\eta_\infty \tag{III.D.12}$$

$$\kappa^2_{t\infty} \equiv \kappa^2_\infty - \gamma^2 \tag{III.D.13}$$

where $\varepsilon_{r\infty}$ is the relative permittivity, $\mu_{r\infty}$ is the relative permeability, and σ_∞ is the conductivity of the region ∞.

Since the region ∞ contains only outwardly traveling waves, the linear combination $p^{(\pm)}_{\nu\infty} \; Z^{(\pm)}_{\nu}(k_o\kappa_{t\infty}\rho)$ becomes

$$p_{\nu\infty} \; H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho) \tag{III.D.14}$$

and the linear combination $q^{(\pm)}_{\nu\infty} \; Z^{(\pm)}_{\nu}(k_o\kappa_{t\infty}\rho)$ becomes

$$q_{\nu\infty} \; H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho) \tag{III.D.15}$$

where $p_{\nu\infty}$ and $q_{\nu\infty}$ are arbitrary constants to be determined later in this study.

In the rest frame of the antenna in the region ∞, the cylindrical components of the field vectors as derived from Eqs. III.B.216 and III.B.217 are

$$E^{\rho}_{\nu\infty} = ik_o^2\gamma\kappa_{t\infty}\, H_\nu^{(1)'}(k_o\kappa_{t\infty}\rho)p_{\nu\infty} + i\omega\mu_o\eta_\infty \frac{i\nu}{\rho} H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)q_{\nu\infty}$$

$$E^{\emptyset}_{\nu\infty} = ik_o\gamma \frac{i\nu}{\rho} H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)p_{\nu\infty} - i\omega\mu_o\eta_\infty k_o\kappa_{t\infty}H_\nu^{(1)'}(k_o\kappa_{t\infty}\rho)q_{\nu\infty}$$

$$E^{z}_{\nu\infty} = k_o^2\kappa_{t\infty}^2 H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)\, p_{\nu\infty} \qquad \text{(III.D.16)}$$

and

$$H^{\rho}_{\nu\infty} = ik_o^2\gamma\kappa_{t\infty}H_\nu^{(1)'}(k_o\kappa_{t\infty}\rho)q_{\nu\infty} - i\omega\varepsilon_o\zeta_\infty \frac{i\nu}{\rho} H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)p_{\nu\infty}$$

$$H^{\emptyset}_{\nu\infty} = ik_o\gamma \frac{i\nu}{\rho} H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)q_{\nu\infty} + i\omega\varepsilon_o\zeta_\infty k_o\kappa_{t\infty}H_\nu^{(1)'}(k_o\kappa_{t\infty}\rho)p_{\nu\infty}$$

$$H^{z}_{\nu\infty} = k_o^2\kappa_{t\infty}^2\, H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)q_{\nu\infty} \qquad \text{(III.D.17)}$$

4. Region i

In the region i ($1 \leq i \leq n$), let

$$\omega'^2_{pi} \equiv \frac{n_i' q^2}{m\varepsilon_o} \qquad \text{(III.D.18)}$$

$$\omega'_{gi} \equiv \frac{q}{m} b'_{oi} \qquad \text{(III.D.19)}$$

$$\zeta'_{\perp i} \equiv 1 - \frac{\omega'^2_{pi}}{\omega'} \frac{\omega' + i\omega'_{ci}}{(\omega' + i\omega'_{ci})^2 - \omega'^2_{gi}} \tag{III.D.20}$$

$$\zeta'_{\parallel i} \equiv 1 - \frac{\omega'^2_{pi}}{\omega'} \frac{1}{\omega' + i\omega'_{ci}} \tag{III.D.21}$$

$$\zeta'_{+i} \equiv \frac{\omega'^{\,2}_{pi}}{\omega'} \frac{\omega'_{gi}}{(\omega' + \omega'_{ci})^2 - \omega'^2_{gi}} \tag{III.D.22}$$

$$\zeta'^2_{ei} \equiv \frac{\zeta'_{\parallel i}}{\zeta'_{\perp i}} (\zeta'_{\perp i} - \gamma'^2) \tag{III.D.23}$$

$$\zeta'^2_{mi} \equiv \frac{\zeta'^2_{\perp i} - \zeta'^2_{+i}}{\zeta'_{\perp i}} - \gamma'^2 \tag{III.D.24}$$

$$\left.\begin{aligned} \eta'_{mi} &\equiv -i\omega'\mu_o k'_o \gamma' \frac{\zeta'_{+i}}{\zeta'_{\perp i}} \qquad &\text{(III.D.25)} \\ \eta'_{ei} &\equiv i\omega'\varepsilon_o k'_o \gamma' \frac{\zeta'_{+i}\zeta'_{\parallel i}}{\zeta'_{\perp i}} \qquad &\text{(III.D.26)} \end{aligned}\right\} \hat{\eta}'_{ei}\,\hat{\eta}'_{mi} \equiv \frac{1}{k_o^4}\,\eta'_{ei}\,\eta'_{mi}$$

$$\hat{\gamma}'_{i\pm} \equiv \frac{\zeta'^2_{ei} - \zeta'^2_{mi}}{2\eta'_{ei}} \left(1 \pm \sqrt{1 + \frac{4\hat{\eta}'_{ei}\,\hat{\eta}'_{mi}}{(\zeta'^2_{ei} - \zeta'^2_{mi})^2}}\right) \tag{III.D.27}$$

$$\gamma'_{i\pm} \equiv k'^2_o\, \hat{\gamma}'_{i\pm} \tag{III.D.28}$$

$$h'^2_{i\pm} \equiv \zeta'^2_{ei} - \hat{\gamma}'_{i\pm}\, \eta'_{mi} \tag{III.B.29}$$

$$\Delta'_i \equiv (\zeta'_{\perp i} - \gamma'^2)^2 - \zeta'^2_{+i} \tag{III.B.30}$$

$$\chi_i' \equiv \frac{1}{1 - (\hat{\gamma}_{i+}'/\hat{\gamma}_{i-}')} \tag{III.D.31}$$

$$\lambda_{i+}' \equiv \frac{i\zeta_{+i}'}{\zeta_{\perp i}' - \gamma'^2} \tag{III.D.32}$$

$$\lambda_{i-}' \equiv \zeta_{\perp i}' - \frac{\zeta_{+i}'^2}{\zeta_{\perp i}' - \gamma'^2} \tag{III.D.33}$$

where n_i' is the electron concentration, ω_{ci}' is the collision frequency, and b_{oi}' is the magnetic bias in the rest frame of the region i .

In the rest frame of the plasma in the region i, the linear combination $p_{\nu i}^{(\pm)'} Z_\nu^{(\pm)}(k_o' h_{i+}' \rho')$ is explicitly chosen to be

$$p_{\nu i+}' J_\nu(k_o' h_{i+}' \rho') + q_{\nu i+}' N_\nu(k_o' h_{i+}' \rho') \tag{III.D.34}$$

Similarly, the linear combination $q_{\nu i}^{(\pm)'} Z_\nu^{(\pm)}(k_o' h_{i-}' \rho')$ is explicitly chosen to be

$$p_{\nu i-}' J_\nu(k_o' h_{i-}' \rho') + q_{\nu i-}' N_\nu(k_o' h_{i-}' \rho') \tag{III.D.35}$$

where $p_{\nu i\pm}'$ and $q_{\nu i\pm}'$ are arbitrary constants to be determined later in the study.

To simplify the notation in what follows, the cylindrical components of the field vectors in the region i are combined as the elements of a single vector $\underline{f}_{\nu i}$, i.e.,

$$\|\underline{f}'_{\nu i}\| = \begin{bmatrix} \underline{E}'_{\nu i} \\ \\ \underline{H}'_{\nu i} \end{bmatrix} \qquad \text{(in s')} \tag{III.D.36}$$

or

$$\|\underline{f}_{\nu i}\| = \begin{bmatrix} \underline{E}_{\nu i} \\ \\ \underline{H}_{\nu i} \end{bmatrix} \qquad \text{(in s)} \tag{III.D.37}$$

Also, the undetermined coefficients are combined as the elements of a vector $\underline{c}_{\nu i}$, i.e.,

$$\|\underline{c}'_{\nu i}\| = \begin{bmatrix} p'_{\nu i+} \\ q'_{\nu i+} \\ p'_{\nu i-} \\ q'_{\nu i-} \end{bmatrix} \qquad \text{(in s')} \tag{III.D.38}$$

$$\|\underline{c}_{\nu i}\| = \begin{bmatrix} p_{\nu i+} \\ \\ q_{\nu i+} \\ \\ p_{\nu i-} \\ \\ q_{\nu i-} \end{bmatrix} \qquad \text{(in s)} \tag{III.D.39}$$

Then, in the rest frame of the plasma in the region i, the cylindrical components of the field vectors as derived from Eqs. III.B.226 and III.B.227 are

$$\underline{f}'_{\nu i} = \underline{\underline{d}}'_{\nu i} \cdot \underline{c}'_{\nu i} \tag{III.D.40}$$

where

$$\|\underline{\underline{d}}'_{\nu i}\| = \begin{bmatrix} a^{\rho'}_{\nu i} & b^{\rho'}_{\nu i} & c^{\rho'}_{\nu i} & d^{\rho'}_{\nu i} \\ a^{\phi'}_{\nu i} & b^{\phi'}_{\nu i} & c^{\phi'}_{\nu i} & d^{\phi'}_{\nu i} \\ a^{z'}_{\nu i} & b^{z'}_{\nu i} & c^{z'}_{\nu i} & d^{z'}_{\nu i} \\ e^{\rho'}_{\nu i} & f^{\rho'}_{\nu i} & g^{\rho'}_{\nu i} & h^{\rho'}_{\nu i} \\ e^{\phi'}_{i} & f^{\phi'}_{i} & g^{\phi'}_{i} & h^{\phi'}_{i} \\ e^{z'}_{\nu i} & f^{z'}_{\nu i} & g^{z'}_{\nu i} & h^{z'}_{\nu i} \end{bmatrix} \qquad \text{(III.D.41)}$$

and

$$a^{\rho'}_{\nu i} \equiv \chi'_i[(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}})\frac{i\nu}{\rho'} J_\nu(k'_o h'_{i+}\rho')$$

$$+(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})k'_o h'_{i+} J'_\nu(k'_o h'_{i+}\rho')]$$

$$b^{\rho'}_{\nu i} \equiv \chi'_i[(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}})\frac{i\nu}{\rho'} N_\nu(k'_o h'_{i+}\rho')$$

$$+(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})k'_o h'_{i+} N'_\nu(k'_o h'_{i+}\rho')]$$

$$c^{\rho'}_{\nu i} \equiv \chi'_i[(-ik'_o\gamma'\lambda'_{i+}\gamma'_{i+}+i\omega'\mu_o)\frac{i\nu}{\rho'} J_\nu(k'_o h'_{i-}\rho')$$

$$+(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})k'_o h'_{i-} J'_\nu(k'_o h'_{i-}\rho')]$$

$$d^{\rho'}_{\nu i} \equiv \chi'_i[(-ik'_o\gamma'\lambda'_{i+}\gamma'_{i+}+i\omega'\mu_o)\frac{i\nu}{\rho'} N_\nu(k'_o h'_{i-}\rho')$$

$$+(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})k'_o h'_{i-} N'_\nu(k'_o h'_{i-}\rho')]$$

$$a^{\phi'}_{\nu i} \equiv \chi'_i[(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+}\frac{1}{\gamma'_{i-}})\frac{i\nu}{\rho'}J_\nu(k'_oh'_{i+}\rho')$$

$$+ (-ik'_o\gamma'\lambda'_{i+} + i\omega'\mu_o\frac{1}{\gamma'_{i-}})k'_oh'_{i+}J'_\nu(k'_oh'_{i+}\rho')]$$

$$b^{\phi'}_{\nu i} \equiv \chi'_i[(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+}\frac{1}{\gamma'_{i-}})\frac{i\nu}{\rho'}N_\nu(k'_oh'_{i+}\rho')$$

$$+ (-ik'_o\gamma'\lambda'_{i+} + i\omega'\mu_o\frac{1}{\gamma'_{i-}})k'_oh'_{i+}N'_\nu(k'_oh'_{i+}\rho')]$$

$$c^{\phi'}_{\nu i} \equiv \chi'_i[(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})\frac{i\nu}{\rho'}J_\nu(k'_oh'_{i-}\rho')$$

$$+ (ik'_o\gamma'\lambda'_{i+}\gamma'_{i+} - i\omega'\mu_o)k'_oh'_{i-}J'_\nu(k'_oh'_{i-}\rho')]$$

$$d^{\phi'}_{\nu i} \equiv \chi'_i[(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})\frac{i\nu}{\rho'}N_\nu(k'_oh'_{i-}\rho')$$

$$+ (ik'_o\gamma'\lambda'_{i+}\gamma'_{i+} - i\omega'\mu_o)k'_oh'_{i-}N'_\nu(k'_oh'_{i-}\rho')]$$

$$a^{z'}_{\nu i} \equiv k'^2_o\frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2}\chi'_i\ J_\nu(k'_oh'_{i+}\rho')$$

$$b^{z'}_{\nu i} \equiv k'^2_o\frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2}\chi'_i\ N_\nu(k'_oh'_{i+}\rho')$$

$$c^{z'}_{\nu i} \equiv -k'^2_o\frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2}\gamma'_{i+}\chi'_i\ J_\nu(k'_oh'_{i-}\rho')$$

$$d^{z'}_{\nu i} \equiv -k'^2_o\frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2}\gamma_{i+}\chi_i\ N_\nu(k'_oh'_{i-}\rho')$$

$$g^{\rho'}_{\nu i} \equiv \chi'_i [(ik'_o\gamma'\lambda'_{i+} + i\omega'\varepsilon_o\lambda'_{i-}\gamma'_{i+}) \frac{i\nu}{\rho'} J_\nu(k'_o h'_{i-}\rho')$$

$$+ (ik'_o\gamma' - i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2\gamma'_{i+}) k'_o h'_{i-} J'_\nu(k'_o h'_{i-}\rho')]$$

$$h^{\rho'}_{\nu i} \equiv \chi'_i [(ik'_o\gamma'\lambda'_{i+} + i\omega'\varepsilon_o\lambda'_{i-}\gamma'_{i+}) \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i-}\rho')$$

$$+ (ik'_o\gamma' - i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2\gamma'_{i+}) k'_o h'_{i-} N'_\nu(k'_o h'_{i-}\rho')]$$

$$e^{\rho'}_{\nu i} \equiv \chi'_i [(-ik'_o\gamma'\lambda'_{i+}\frac{1}{\gamma'_{i-}} - i\omega'\varepsilon_o\lambda'_{i-}) \frac{i\nu}{\rho'} J_\nu(k'_o h'_{i+}\rho')$$

$$+(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) k'_o h'_{i+} J'_\nu(k'_o h'_{i+}\rho')]$$

$$f^{\rho'}_{\nu i} \equiv \chi'_i [(-ik'_o\gamma'\lambda'_{i+}\frac{1}{\gamma'_{i-}} - i\omega'\varepsilon_o\lambda'_{i-}) \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i+}\rho')$$

$$+(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) k'_o h'_{i+} N'_\nu(k'_o h'_{i+}\rho')]$$

$$g^{\phi'}_{\nu i} \equiv \chi'_i [(ik'_o\gamma' - i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2\gamma'_{i+}) \frac{i\nu}{\rho'} J_\nu(k'_o h'_{i-}\rho')$$

$$+ (-ik'_o\gamma'\lambda'_{i+} - i\omega'\varepsilon_o\lambda'_{i-}\gamma'_{i+}) k'_o h'_{i-} J'_\nu(k'_o h'_{i-}\rho')$$

$$h^{\phi'}_{\nu i} \equiv \chi'_i [(ik'_o\gamma' - i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2\gamma'_{i+}) \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i-}\rho')$$

$$+ (-ik'_o\gamma'\lambda'_{i+} - i\omega'\varepsilon_o\lambda'_{i-}\gamma'_{i+}) k'_o h'_{i-} N'_\nu(k'_o h'_{i-}\rho')$$

$$e^{\phi'}_{\nu i} \equiv \chi'_i [(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) \frac{i\nu}{\rho'} J_\nu(k'_o h'_{i+}\rho')$$

$$+ (ik'_o\gamma'\lambda'_{i+}\frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i-}) k'_o h'_{i+} J'_\nu(k'_o h'_{i+}\rho')]$$

$$f^{\phi'}_{\nu i} \equiv \chi'_i [(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i+}\rho')$$

$$+ (ik'_o\gamma'\lambda'_{i+} \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i-})\, k'_o h'_{i+}\, N'_\nu(k'_o h'_{i+}\rho')]$$

$$g^{z'}_{\nu i} \equiv k'^2_o \frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2} \chi'_i\, J_\nu(k'_o h'_{i-}\rho')$$

$$h^{z'}_{\nu i} \equiv k'^2_o \frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2} \chi'_i\, N_\nu(k'_o h'_{i-}\rho')$$

$$e^{z'}_{\nu i} \equiv -k'^2_o \frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2} \frac{1}{\gamma'_{i-}} \chi'_i\, J_\nu(k'_o h'_{i+}\rho')$$

$$f^{z'}_{\nu i} \equiv -k'^2_o \frac{\Delta'_i}{\zeta'_{\perp i} - \gamma'^2} \frac{1}{\gamma'_{i-}} \chi'_i\, N_\nu(k'_o h'_{i+}\rho') \qquad \text{(III.D.42)}$$

By using the constitutive relations developed in the previous section of this study, one can write the cylidrical components of the field vectors in the rest frame of the antenna in the following form:

$$\underline{f}_{\nu i} = \underline{\underline{d}}_{\nu i} \cdot \underline{c}_{\nu i} \qquad \text{(III.D.43)}$$

where

$$\|\underline{\underline{d}}_{\nu i}\| = \begin{bmatrix} a^{\rho}_{\nu i} & b^{\rho}_{\nu i} & c^{\rho}_{\nu i} & d^{\rho}_{\nu i} \\ a^{\phi}_{\nu i} & b^{\phi}_{\nu i} & c^{\phi}_{\nu i} & d^{\phi}_{\nu i} \\ a^{z}_{\nu i} & b^{z}_{\nu i} & c^{z}_{\nu i} & d^{z}_{\nu i} \\ e^{\rho}_{\nu i} & f^{\rho}_{\nu i} & g^{\rho}_{\nu i} & h^{\rho}_{\nu i} \\ e^{\phi}_{\nu i} & f^{\phi}_{\nu i} & g^{\phi}_{\nu i} & h^{\phi}_{\nu i} \\ e^{z}_{\nu i} & f^{z}_{\nu i} & g^{z}_{\nu i} & h^{z}_{\nu i} \end{bmatrix} \tag{III.D.44}$$

and

$$a^{\rho}_{\nu i} \equiv [\gamma_z(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}}) + \mu_o\beta_z(-ik'_o\gamma' \frac{1}{\gamma'_{i-}}$$

$$+ i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2)]\chi'_i \frac{i\nu}{\rho'} J_\nu(k'_oh'_{i+}\rho')$$

$$+ [\gamma_z(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}}) + \mu_o\beta_z(ik'_o\gamma'\lambda'_{i+} \frac{1}{\gamma'_{i-}}$$

$$+ i\omega'\varepsilon_o\lambda'_{i-})]\chi'_i k'_oh'_{i+}J'_\nu(k'_oh'_{i+}\rho')$$

$$b^{\rho}_{\nu i} \equiv [\gamma_z(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}}) + \mu_o\beta_z(-ik'_o\gamma' \frac{1}{\gamma'_{i-}}$$

$$+ i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2)]\chi'_i \frac{i\nu}{\rho'} N_\nu(k'_oh'_{i+}\rho')$$

$$+ [\gamma_z(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}}) + \mu_o\beta_z(ik'_o\gamma'\lambda'_{i+} \frac{1}{\gamma'_{i-}}$$

$$+ i\omega'\varepsilon_o\lambda'_{i-})]\chi'_i k'_oh'_{i+}N'_\nu(k'_oh'_{i+}\rho')$$

$$c^{\rho}_{\nu i} \equiv [\gamma_z(-ik_o'\gamma'\lambda_{i+}'\gamma_{i+}'+i\omega'\mu_o)+\mu_o\beta_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}')]$$

$$\times\ \chi_i'\frac{i\nu}{\rho'}J_\nu(k_o'h_{i-}'\rho')$$

$$+\ [\gamma_z(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')+\mu_o\beta_z(-ik_o'\gamma'\lambda_{i+}' - i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}')]$$

$$\times\ \chi_i'k_o'h_{i-}'\ J_\nu'(k_o'h_{i-}'\rho')$$

$$d^{\rho}_{\nu i} \equiv [\gamma_z(-ik_o'\gamma'\lambda_{i+}'\gamma_{i+}'+i\omega'\mu_o)+\mu_o\beta_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}')]$$

$$\times\ \chi_i'\frac{i\nu}{\rho'}N_\nu(k_o'h_{i-}'\rho')$$

$$+\ [\gamma_z(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')+\mu_o\beta_z(-ik_o'\gamma'\lambda_{i+}' - i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}')]$$

$$\times\ \chi_i'k_o'h_{i-}'\ N_\nu'(k_o'h_{i-}'\rho')$$

$$a^{\emptyset}_{\nu i} \equiv [\gamma_z(ik_o'\gamma'+i\omega'\mu_o\lambda_{i+}'\frac{1}{\gamma_{i-}'}) - \mu_o\beta_z(-ik_o'\gamma'\lambda_{i+}'\frac{1}{\gamma_{i-}'} - i\omega'\varepsilon_o\lambda_{i-}')]$$

$$\times\ \chi_i'\frac{i\nu}{\rho'}J_\nu(k_o'h_{i+}'\rho')$$

$$+\ [\gamma_z(-ik_o'\gamma'\lambda_{i+}'+i\omega'\mu_o\frac{1}{\gamma_{i-}'}) - \mu_o\beta_z(-ik_o'\gamma'\frac{1}{\gamma_{i-}'}+i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2)]$$

$$\times\ \chi_i'k_o'h_{i+}'J_\nu'(k_o'h_{i+}'\rho')$$

$$b^{\emptyset}_{\nu i} \equiv [\gamma_z(ik_o'\gamma' + i\omega'\mu_o\lambda_{i+}' \frac{1}{\gamma_{i-}'}) - \mu_o\beta_z(-ik_o'\gamma'\lambda_{i+}' \frac{1}{\gamma_{i-}'} - i\omega'\varepsilon_o\lambda_{i-}')]$$

$$\times \; \chi_i' \frac{i\nu}{\rho'} N_\nu(k_o'h_{i+}'\rho')$$

$$+ \; [\gamma_z(-ik_o'\gamma'\lambda_{i+}' + i\omega'\mu_o \frac{1}{\gamma_{i-}'}) - \mu_o\beta_z(-ik_o'\gamma' \frac{1}{\gamma_{i-}'} + i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2)]$$

$$\times \; \chi_i'k_o'h_{i+}'N_\nu'(k_o'h_{i+}'\rho')$$

$$c^{\emptyset}_{\nu i} \equiv [\gamma_z(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}') - \mu_o\beta_z(ik_o'\gamma'\lambda_{i+}' + i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}')]$$

$$\times \; \chi_i' \frac{i\nu}{\rho'} J_\nu(k_o'h_{i-}'\rho')$$

$$+ \; [\gamma_z(ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' - i\omega'\mu_o) - \mu_o\beta_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}')]$$

$$\times \; \chi_i'k_o'h_{i-}' \; J_\nu'(k_o'h_{i-}'\rho')$$

$$d^{\emptyset}_{\nu i} \equiv [\gamma_z(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}') - \mu_o\beta_z(ik_o'\gamma'\lambda_{i+}' + i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}')]$$

$$\times \; \chi_i' \frac{i\nu}{\rho'} N_\nu(k_o'h_{i-}'\rho')$$

$$+ \; [\gamma_z(ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' - i\omega'\mu_o) - \mu_o\beta_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}')]$$

$$\times \; \chi_i'k_o'h_{i-}' \; N_\nu'(k_o'h_{i-}'\rho')$$

$$a^z_{\nu i} \equiv k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \chi_i' \, J_\nu(k_o' h_{i+}' \rho')$$

$$b^z_{\nu i} \equiv k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \chi_i' \, N_\nu(k_o' h_{i+}' \rho')$$

$$c^z_{\nu i} \equiv -k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \gamma_{i+}' \chi_i' J_\nu(k_o' h_{i-}' \rho')$$

$$d^z_{\nu i} \equiv -k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \gamma_{i+}' \chi_i' N_\nu(k_o' h_{i-}' \rho')$$

$$g^\rho_{\nu i} \equiv \{\gamma_z(ik_o'\gamma'\lambda_{i+}' + i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')$$

$$+ i\zeta_{+i}'(-ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' + i\omega'\mu_o)]\} \chi_i' \frac{i\nu}{\rho'} J_\nu(k_o' h_{i-}' \rho')$$

$$+\{\gamma_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' - i\omega'\mu_o)$$

$$+ i\zeta_{+i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')]\} \chi_i' k_o' h_{i-}' \, J_\nu'(k_o' h_{i-}' \rho')$$

$$h^\rho_{\nu i} \equiv \{\gamma_z(ik_o'\gamma'\lambda_{i+}' + i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')$$

$$+ i\zeta_{+i}'(-ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' + i\omega'\mu_o)]\} \chi_i' \frac{i\nu}{\rho'} N_\nu(k_o' h_{i-}' \rho')$$

$$+\{\gamma_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' - i\omega'\mu_o)$$

$$+ i\zeta_{+i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')]\} \chi_i' k_o' h_{i-}' \, N_\nu'(k_o' h_{i-}' \rho')$$

$$e^{\rho}_{\nu i} \equiv \{\gamma_z(-ik_o'\gamma'\lambda_{i+}' \frac{1}{\gamma_{i-}'} - i\omega'\varepsilon_o\lambda_{i-}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(ik_o'\gamma' + i\omega'\mu_o\lambda_{i+}' \frac{1}{\gamma_{i-}'})$$

$$+i\zeta_{+i}'(ik_o'\gamma'\lambda_{i+}' - i\omega'\mu_o \frac{1}{\gamma_{i-}'})]\}\chi_i' \frac{i\nu}{\rho'} J_\nu(k_o'h_{i+}'\rho')$$

$$+\{\gamma_z(-ik_o'\gamma' \frac{1}{\gamma_{i-}'} + i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2) - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(-ik_o'\gamma'\lambda_{i+}' + i\omega'\mu_o \frac{1}{\gamma_{i-}'})$$

$$+i\zeta_{+i}'(ik_o'\gamma' + i\omega'\mu_o\lambda_{i+}' \frac{1}{\gamma_{i-}'})]\}\chi_i'k_o'h_{i+}' J_\nu'(k_o'h_{i+}'\rho')$$

$$f^{\rho}_{\nu i} \equiv \{\gamma_z(-ik_o'\gamma'\lambda_{i+}' \frac{1}{\gamma_{i-}'} - i\omega'\varepsilon_o\lambda_{i-}') - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(ik_o'\gamma' + i\omega'\mu_o\lambda_{i+}' \frac{1}{\gamma_{i-}'})$$

$$+i\zeta_{+i}'(ik_o'\gamma'\lambda_{i+}' - i\omega'\mu_o \frac{1}{\gamma_{i-}'})]\}\chi_i' \frac{i\nu}{\rho'} N_\nu(k_o'h_{i+}'\rho')$$

$$+\{\gamma_z(-ik_o'\gamma' \frac{1}{\gamma_{i-}'} + i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2) - \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(-ik_o'\gamma'\lambda_{i+}' + i\omega'\mu_o \frac{1}{\gamma_{i-}'})$$

$$+i\zeta_{+i}'(ik_o'\gamma' + i\omega'\mu_o\lambda_{i+}' \frac{1}{\gamma_{i-}'})]\}\chi_i'k_o'h_{i+}' N_\nu'(k_o'h_{i+}'\rho')$$

$$g^{\phi}_{\nu i} \equiv \{\gamma_z(ik_o'\gamma' - i\omega'\varepsilon_o\lambda_{i+}'\gamma'^2\gamma_{i+}') + \varepsilon_o\beta_z\gamma_z[\zeta_{\perp i}'(-ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' + i\omega'\mu_o)$$

$$-i\zeta_{+i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')]\} \chi_i' \frac{i\nu}{\rho'} J_\nu(k_o'h_{i-}'\rho')$$

$$+\{\gamma_z(-ik_o'\gamma'\lambda_{i+}' - i\omega'\varepsilon_o\lambda_{i-}'\gamma_{i+}') + \varepsilon_o\beta_z\gamma_z(\zeta_{\perp i}'(-ik_o'\gamma'\gamma_{i+}' - i\omega'\mu_o\lambda_{i+}')$$

$$-i\zeta_{+i}'(ik_o'\gamma'\lambda_{i+}'\gamma_{i+}' - i\omega'\mu_o)]\} \chi_i' k_o'h_{i-}' J_\nu'(k_o'h_{i-}'\rho')$$

$$h^{\emptyset}_{\nu i} \equiv \{\gamma_z(ik'_o\gamma' - i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2\gamma'_{i+}) + \varepsilon_o\beta_z\gamma_z[-\zeta'_{\perp i}(-ik'_o\gamma'\lambda'_{i+}\gamma'_{i+} + i\omega'\mu_o)$$

$$- i\zeta'_{+i}(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})]\chi'_i \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i-}\rho')$$

$$+\{\gamma_z(-ik'_o\gamma'\lambda'_{i+} - i\omega'\varepsilon_o\lambda'_{i-}\gamma'_{i+}) + \varepsilon_o\beta_z\gamma_z(\zeta'_{\perp i}(-ik'_o\gamma'\gamma'_{i+} - i\omega'\mu_o\lambda'_{i+})$$

$$- i\zeta'_{+i}(ik'_o\gamma'\lambda'_{i+}\gamma'_{i+} - i\omega'\mu_o)]\}\chi'_i k'_o h'_{i-} N'_\nu(k'_o h'_{i-}\rho')$$

$$e^{\emptyset}_{\nu i} \equiv \{\gamma_z(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) + \varepsilon_o\beta_z\gamma_z[\zeta'_{\perp i}(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}})$$

$$-i\zeta'_{+i}(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})]\}\chi'_i \frac{i\nu}{\rho'} J_\nu(k'_o h'_{i+}\rho')$$

$$+\{\gamma_z(ik'_o\gamma'\lambda'_{i+} \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i-}) + \varepsilon_o\beta_z\gamma_z[\zeta'_{\perp i}(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})$$

$$-i\zeta'_{+i}(-ik'_o\gamma'\lambda'_{i+} + i\omega'\mu_o \frac{1}{\gamma'_{i-}})]\}\chi'_i\ k'_o h'_{i+}\ J'_\nu(k'_o h'_{i+}\rho')$$

$$f^{\emptyset}_{\nu i} \equiv \{\gamma_z(-ik'_o\gamma' \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i+}\gamma'^2) + \varepsilon_o\beta_z\gamma_z[\zeta'_{\perp i}(ik'_o\gamma'\lambda'_{i+} - i\omega'\mu_o \frac{1}{\gamma'_{i-}})$$

$$-i\zeta'_{+i}(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})]\}\chi'_i \frac{i\nu}{\rho'} N_\nu(k'_o h'_{i+}\rho')$$

$$+\{\gamma_z(ik'_o\gamma'\lambda'_{i+} \frac{1}{\gamma'_{i-}} + i\omega'\varepsilon_o\lambda'_{i-}) + \varepsilon_o\beta_z\gamma_z[\zeta'_{\perp i}(ik'_o\gamma' + i\omega'\mu_o\lambda'_{i+} \frac{1}{\gamma'_{i-}})$$

$$-i\zeta'_{+i}(-ik'_o\gamma'\lambda'_{i+} + i\omega'\mu_o \frac{1}{\gamma'_{i-}})]\}\chi'_i\ k'_o h'_{i+}\ N'_\nu(k'_o h'_{i+}\rho')$$

$$g^{z}_{\nu i} \equiv k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \chi_i' \; J_\nu(k_o' h_{i-}' \rho')$$

$$h^{z}_{\nu i} \equiv k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \chi_i' \; N_\nu(k_o' h_{i-}' \rho')$$

$$e^{z}_{\nu i} \equiv -k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \frac{1}{\gamma_{i-}'} \chi_i' \; J_\nu(k_o' h_{i+}' \rho')$$

$$f^{z}_{\nu i} \equiv -k_o'^2 \frac{\Delta_i'}{\zeta_{\perp i}' - \gamma'^2} \frac{1}{\gamma_{i-}'} \chi_i' \; N_\nu(k_o' h_{i+}' \rho') \qquad \text{(III.D.45)}$$

The primed quantities appearing in the above equations can be removed by using the following transformations between ω and $k_o\gamma$:

$$\omega' = \gamma_z(\omega - \beta_z k_o \gamma)$$

$$k_o'\gamma' = \gamma_z(k_o\gamma - \beta_z \omega) \qquad \text{(III.D.46)}$$

This transformation is easily accomplished, but because the resulting representations of the field vectors become very unwieldy, the results of this transformation are not shown. In what follows, it will be remembered that any primed variable is reducible to an equivalent combination of unprimed variables through the use of Eq. III.D.45.

E. Boundary Conditions

1. Introduction and Notation

The boundary conditions on the field vectors in the rest frame of the antenna are simply that the tangential electric and magnetic field vectors are continuous across each layer of the plasma. Therefore E^{ϕ} and E^{z}, H^{ϕ} and H^{z} must vary continuously across each interface of the plasma.

To simplify the notation in what follows, in the region 0 , let

$$||\underline{\underline{m}}_{\nu o}|| = \begin{bmatrix} a^{\phi}_{\nu o} & 0 & c^{\phi}_{\nu o} & 0 \\ a^{z}_{\nu o} & 0 & 0 & 0 \\ e^{\phi}_{\nu o} & 0 & g^{\phi}_{\nu o} & 0 \\ 0 & 0 & g^{z}_{\nu o} & 0 \end{bmatrix} \qquad \text{(III.E.1)}$$

where

$$\begin{aligned} a^{\phi}_{\nu o} &\equiv ik_o\gamma \frac{i\nu}{\rho} J_\nu(k_o\kappa_{to}\rho) \\ a^{z}_{\nu o} &\equiv k_o^2\kappa_{to}^2 \; J_\nu(k_o\kappa_{to}\rho) \\ c^{\phi}_{\nu o} &\equiv -i\omega\mu_o\eta_o k_o\kappa_{to} \; J'_\nu(k_o\kappa_{to}\rho) \\ e^{\phi}_{\nu o} &\equiv i\omega\varepsilon_o\zeta_o k_o\kappa_{to} \; J'_\nu(k_o\kappa_{to}\rho) \\ g^{\phi}_{\nu o} &\equiv ik_o\gamma \frac{i\nu}{\rho} J_\nu(k_o\kappa_{to}\rho) \\ g^{z}_{\nu o} &\equiv k_o^2\kappa_{to}^2 \; J_\nu(k_o\kappa_{to}\rho) \end{aligned} \qquad \text{(III.E.2)}$$

and

$$\|\underline{k}_o\| = \begin{bmatrix} e^{\emptyset} \\ e^{z} \\ h^{\emptyset} \\ h^{z} \end{bmatrix} \tag{III.E.3}$$

In the region i , let

$$\|\underline{\underline{m}}_{\nu i}\| = \begin{bmatrix} a^{\emptyset}_{\nu i} & b^{\emptyset}_{\nu i} & c^{\emptyset}_{\nu i} & d^{\emptyset}_{\nu i} \\ a^{z}_{\nu i} & b^{z}_{\nu i} & c^{z}_{\nu i} & d^{z}_{\nu i} \\ e^{\emptyset}_{\nu i} & f^{\emptyset}_{\nu i} & g^{\emptyset}_{\nu i} & h^{\emptyset}_{\nu i} \\ e^{z}_{\nu i} & f^{z}_{\nu i} & g^{z}_{\nu i} & h^{z}_{\nu i} \end{bmatrix} \tag{III.E.4}$$

In the region ∞ , let

$$\|\underline{\underline{m}}_{\nu\infty}\| = \begin{bmatrix} a^{\emptyset}_{\nu\infty} & 0 & c^{\emptyset}_{\nu\infty} & 0 \\ a^{z}_{\nu\infty} & 0 & 0 & 0 \\ e^{\emptyset}_{\nu\infty} & 0 & g^{\emptyset}_{\nu\infty} & 0 \\ 0 & 0 & g^{z}_{\nu\infty} & 0 \end{bmatrix} \tag{III.E.5}$$

where

$$a^{\emptyset}_{\nu\infty} \equiv ik_o\gamma \frac{i\nu}{\rho} H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho)$$

$$a^{z}_{\nu\infty} \equiv k_o^2\kappa_{t\infty}^2 H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho)$$

$$c^{\emptyset}_{\nu\infty} \equiv -i\omega\mu_o\eta_\infty k_o\kappa_{t\infty} H^{(1)\prime}_{\nu}(k_o\kappa_{t\infty}\rho)$$

$$e^{\emptyset}_{\nu\infty} \equiv i\omega\varepsilon_o\zeta_\infty k_o\kappa_{t\infty} H^{(1)\prime}_{\nu}(k_o\kappa_{t\infty}\rho)$$

$$g^{\phi}_{\nu\infty} \equiv ik_o\gamma \frac{i\nu}{\rho} H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho)$$

$$g^{z}_{\nu\infty} \equiv k_o^2\kappa_{t\infty}^2 H^{(1)}_{\nu}(k_o\kappa_{t\infty}\rho) \qquad \text{(III.E.6)}$$

Also, the tangential field vectors in the various regions are grouped as follows:

$$\|\underline{t}_{\nu o}\| = \begin{bmatrix} E^{\phi o} \\ E^{zo} \\ H^{\phi o} \\ H^{zo} \end{bmatrix} \qquad \|\underline{t}_{\nu i}\| = \begin{bmatrix} E^{\phi i} \\ E^{zi} \\ H^{\phi i} \\ H^{zi} \end{bmatrix} \qquad \|\underline{t}_{\nu\infty}\| = \begin{bmatrix} E^{\phi\infty} \\ E^{z\infty} \\ H^{\phi\infty} \\ H^{z\infty} \end{bmatrix} \qquad \text{(III.E.7)}$$

Similarly, the undetermined coefficients in the various regions are grouped as follows:

$$\|\underline{c}_{\nu o}\| = \begin{bmatrix} p_{\nu o} \\ 0 \\ q_{\nu o} \\ 0 \end{bmatrix} \qquad \|\underline{c}_{\nu i}\| = \begin{bmatrix} p_{\nu i+} \\ q_{\nu i+} \\ p_{\nu i-} \\ q_{\nu i-} \end{bmatrix} \qquad \|\underline{c}_{\nu\infty}\| = \begin{bmatrix} p_{\nu\infty} \\ 0 \\ q_{\nu\infty} \\ 0 \end{bmatrix} \qquad \text{(III.E.8)}$$

2. Matrix Solution

In terms of the symbols just introduced , the following equations can be written

$$\underline{t}_{\nu o} = \underline{\underline{m}}_{\nu o} \cdot \underline{c}_{\nu o} + \underline{k}_o \qquad \text{(III.E.9)}$$

$$\underline{t}_{\nu i} = \underline{\underline{m}}_{\nu i} \cdot \underline{c}_{\nu i} \qquad \text{(III.E.10)}$$

$$\underline{t}_{\nu\infty} = \underline{\underline{m}}_{\nu\infty} \cdot \underline{c}_{\nu\infty} \qquad \text{(III.E.11)}$$

Therefore, the boundary conditions at $\rho = \rho_o$ are

$$\underline{\underline{m}}_{\nu o}\Big|_{\rho=\rho_o} \cdot \underline{c}_{\nu o} + \underline{k}_o\Big|_{\rho=\rho_o} = \underline{\underline{m}}_{\nu 1}\Big|_{\rho=\rho_o} \cdot \underline{c}_{\nu 1} \qquad \text{(III.E.12)}$$

at $\rho = \rho_i$, the boundary conditions are

$$\underline{\underline{m}}_{\nu i}\Big|_{\rho=\rho_i} \cdot \underline{c}_{\nu i} = \underline{\underline{m}}_{\nu,i+1}\Big|_{\rho=\rho_i} \cdot \underline{c}_{\nu,i+1} \qquad \text{(III.E.13)}$$

and, at $\rho = \rho_n = \rho_\infty$, the boundary conditions are

$$\underline{\underline{m}}_{\nu n}\Big|_{\rho=\rho_n} \cdot \underline{c}_{\nu n} = \underline{\underline{m}}_{\nu\infty}\Big|_{\rho=\rho_\infty} \cdot \underline{c}_{\nu\infty} \qquad \text{(III.E.14)}$$

These equations can be summarized by stating

$$\underline{\underline{m}}_{\nu o}\Big|_{\rho=\rho_o} \cdot \underline{c}_{\nu o} + \underline{k}_o\Big|_{\rho=\rho_o} = \underline{\underline{m}}_{\nu 1}\Big|_{\rho=\rho_o} \cdot \Big\{ \prod_{i=1}^{n-1} (\underline{\underline{m}}_{\nu i})^{-1}\Big|_{\rho=\rho_i} \cdot \underline{\underline{m}}_{\nu,i+1}\Big|_{\rho=\rho_i} \Big\}$$
$$\cdot (\underline{\underline{m}}_{\nu n})^{-1}\Big|_{\rho=\rho_n} \cdot \underline{\underline{m}}_{\nu\infty}\Big|_{\rho=\rho_\infty} \cdot \underline{c}_{\nu\infty} \qquad (n \neq 1) \qquad \text{(III.E.15)}$$

Let

$$\underline{\underline{m}}_{\nu} \equiv \underline{\underline{m}}_{\nu 1}\Big|_{\rho=\rho_o} \cdot \Big\{ \prod_{i=1}^{n-1} (\underline{\underline{m}}_{\nu i})^{-1}\Big|_{\rho=\rho_i} \cdot \underline{\underline{m}}_{\nu,i+1}\Big|_{\rho=\rho_i} \Big\} \cdot (\underline{\underline{m}}_{\nu n})^{-1}\Big|_{\rho=\rho_n}$$
$$(n \neq 1) \qquad \text{(III.E.16)}$$

then equation III.E.15 reduces to the simpler form

$$\underline{\underline{m}}_{\nu o}\Big|_{\rho=\rho_o} \cdot \underline{c}_{\nu o} - \underline{\underline{m}}_{\nu} \cdot \underline{\underline{m}}_{\nu\infty}\Big|_{\rho=\rho_\infty} \cdot \underline{c}_{\nu\infty} = - \underline{k}_o\Big|_{\rho=\rho_o} \qquad \text{(III.E.17)}$$

This system of equations can be easily solved for the remaining constants $p_{\nu 0}$, $q_{\nu 0}$, $p_{\nu\infty}$, and $q_{\nu\infty}$, by elementary matrix theory.

F. Asymptotic Expansion of the Radiation Fields

The integral expressions for the cylindrical components of the field vectors in the region ∞ are

$$E^{\rho} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [ik_o^2\gamma\kappa_{t\infty}p_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho) + i\omega\mu_o\eta_{\infty}\frac{i\nu}{\rho}q_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)]e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}$$

$$E^{\phi} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [ik_o\gamma\frac{i\nu}{\rho}p_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho) - i\omega\mu_o\eta_{\infty}k_o\kappa_{t\infty}q_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)]e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}$$

$$E^{z} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [k_o^2\kappa_{t\infty}^2p_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)]e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}$$

and

$$H^{\rho} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [ik_o^2\gamma\kappa_{t\infty}q_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho) - i\omega\varepsilon_o\zeta_{\infty}\frac{i\nu}{\rho}p_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)]e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}$$

$$H^{\phi} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [ik_o\gamma\frac{i\nu}{\rho}q_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho) + i\omega\varepsilon_o\zeta_{\infty}k_o\kappa_{t\infty}p_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)]e^{i\nu(\phi-\phi_s)}e^{ik_o\gamma(z-z_s)}$$

$$H^z = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [k_o^2\kappa_{t\infty}^2 q_{\nu\infty} H_\nu^{(1)}(k_o\kappa_{t\infty}\rho)] e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)} \tag{III.F.2}$$

The constants $p_{\nu\infty}$ and $q_{\nu\infty}$ are now known from the work performed in the previous section of this study.

If the region ∞ is assumed to be nonconducting,

$$\sigma_\infty = 0 \tag{III.F.3}$$

Then κ_∞ reduces to the real number

$$\kappa_\infty = \sqrt{\zeta_\infty \kappa_\infty} = \sqrt{\mu_{r\infty}\varepsilon_{r\infty}} \tag{III.F.4}$$

The integration in the complex γ plane is along the real axis of γ from $-\infty$ to $+\infty$ with an indentation below the branch point at $\gamma = +\kappa_\infty$ and above the branch point at $\gamma = -\kappa_\infty$, as shown in Figure 16. No indentations are required if κ_∞ is allowed to have a vanishingly small but finite positive imaginary part corresponding to the presence of some conductivity in the region ∞.

In the present case, it is assumed that $k_o\kappa_{t\infty}\rho >> 1$. This corresponds to evaluating the fields in the radiation zone of the antenna, since the distance from the point of observation to the outer radius of the cylinder is large compared with the wavelength. With this assumption, the radiation fields, correct to order $1/\rho$, reduce to the expressions

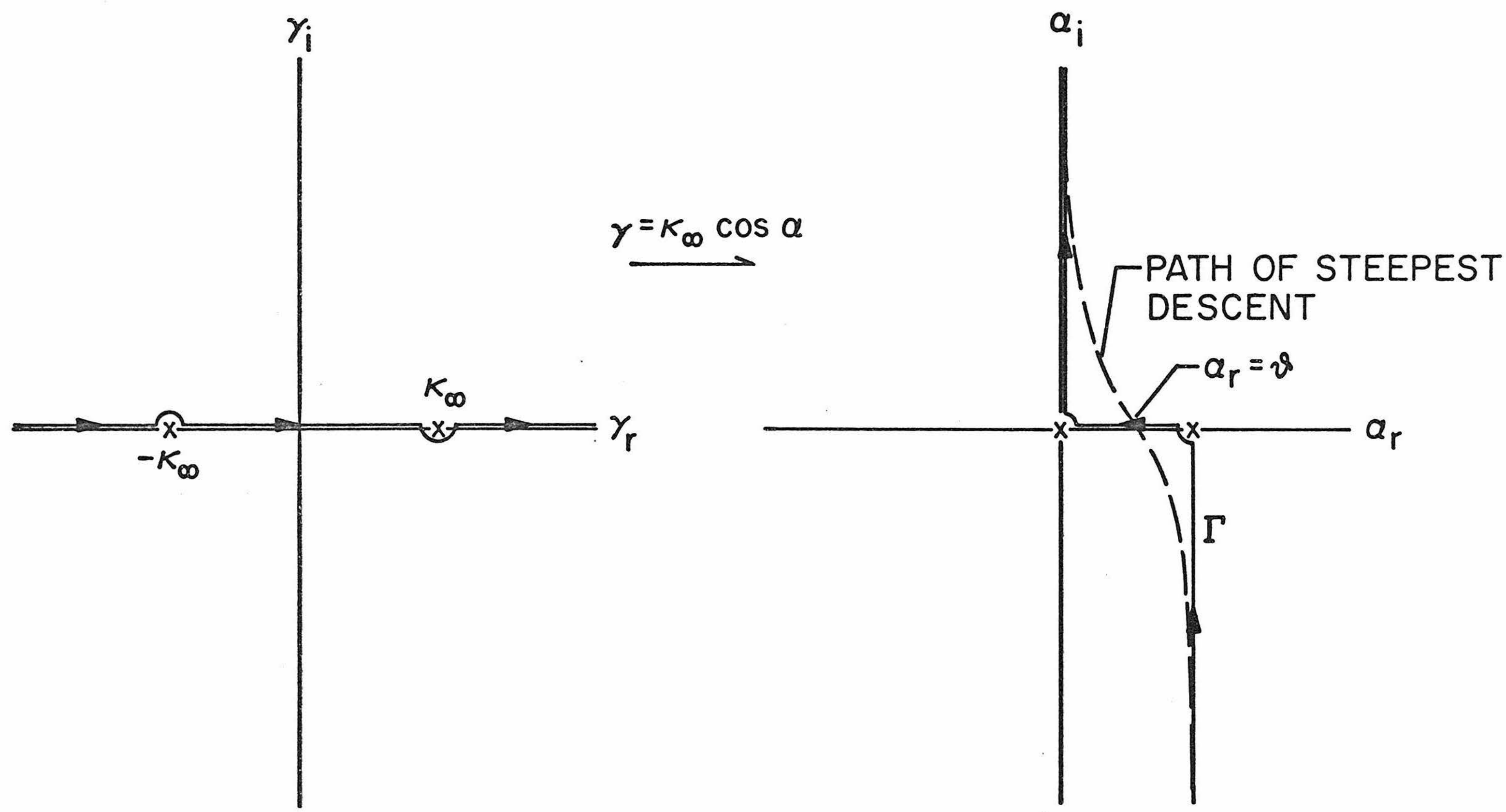


Fig.16 Integration Paths

$$E^{\rho} \simeq \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, ik_o^2\gamma\kappa_{t\infty}p_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)}$$

$$E^{\phi} \simeq -\sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, i\omega\mu_o\eta_{\infty}k_o\kappa_{t\infty}q_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)}$$

$$E^{z} \simeq \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, k_o^2\kappa_{t\infty}^2p_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)}$$

and

$$H^{\rho} \simeq \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, ik_o^2\gamma\kappa_{t\infty}q_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)}$$

$$H^{\phi} \simeq \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, i\omega\varepsilon_o\zeta_{\infty}k_o\kappa_{t\infty}p_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)}$$

$$H^{z} \simeq \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, k_o^2\kappa_{t\infty}^2q_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)} e^{ik_o\gamma(z-z_s)} \qquad \text{(III.F.6)}$$

If the spherical transformation

$$\begin{aligned} \rho &= r\sin\theta & v^r &= v^{\rho}\sin\theta + v^z\cos\theta \\ \phi &= \phi & v^{\theta} &= v^{\rho}\cos\theta - v^z\sin\theta \\ z &= r\cos\theta & v^{\phi} &= v^{\phi} \end{aligned} \qquad \text{(III.F.7)}$$

is performed, the resulting theta and phi components of the radiation field are given by

$$E^{\theta} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} [ik_o^2\gamma\kappa_{t\infty}p_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)\cos\theta$$

$$- k_o^2\kappa_{t\infty}^2p_{\nu\infty}H_{\nu}^{(1)}(k_o\kappa_{t\infty}\rho)\sin\theta]e^{i\nu(\phi-\phi_s)}\ e^{ik_o\gamma(z-z_s)}$$

$$E^{\phi} = -\sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}}\ i\omega\mu_o\eta_\infty k_o\kappa_{t\infty}q_{\nu\infty}H_{\nu}^{(1)'}(k_o\kappa_{t\infty}\rho)e^{i\nu(\phi-\phi_s)}\ e^{ik_o\gamma(z-z_s)} \tag{III.F.8}$$

and

$$H^{\theta} = -\frac{k_o\kappa_\infty}{\omega\mu_o\eta_\infty}E^{\phi}$$

$$H^{\phi} = \frac{k_o\kappa_\infty}{\omega\mu_o\eta_\infty}E^{\theta} \tag{III.F.9}$$

Also, under the assumption that $k_o\kappa_{t\infty}\rho >> 1$, it is permissible to replace the Hankel function and its derivative by the first term of its asymptotic expansions (16)

$$H_{\nu}^{(1)}(k_o\kappa_{t\infty}r\sin\theta) \simeq \sqrt{\frac{2}{\pi k_o\kappa_{t\infty}r\sin\theta}}\ e^{ik_o\kappa_{t\infty}r\sin\theta}\ e^{-i\nu\frac{\pi}{2}}\ e^{-i\frac{\pi}{4}} \tag{III.F.10}$$

$$H_{\nu}^{(1)'}(k_o\kappa_{t\infty}r\sin\theta) \simeq i\sqrt{\frac{2}{\pi k_o\kappa_{t\infty}r\sin\theta}}\ e^{ik_o\kappa_{t\infty}r\sin\theta}\ e^{-i\nu\frac{\pi}{2}}\ e^{-i\frac{\pi}{4}} \tag{III.F.11}$$

When these expansions are used the theta and phi components of the electric field vectors become

$$E^{\theta} = -\sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \; k_o^2(\gamma\kappa_{t\infty}\cos\theta + \kappa_{t\infty}^2 \sin\theta) p_{\nu\infty} \sqrt{\frac{2}{\pi k_o \kappa_{t\infty} r \sin\theta}}$$

$$\times \; e^{ik_o\kappa_{t\infty} r \sin\theta} \; e^{-i\nu\frac{\pi}{2}} \; e^{-i\frac{\pi}{4}} \; e^{i\nu(\phi-\phi_s)} \; e^{ik_o\gamma(z-z_s)}$$

$$E^{\phi} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \; \omega\mu_o\eta_\infty k_o\kappa_{t\infty} q_{\nu\infty} \sqrt{\frac{2}{\pi k_o \kappa_{t\infty} r \sin\theta}}$$

$$\times \; e^{ik_o\kappa_{t\infty} r \sin\theta} \; e^{-i\nu\frac{\pi}{2}} \; e^{-i\frac{\pi}{4}} \; e^{i\nu(\phi-\phi_s)} \; e^{ik_o\gamma(z-z_s)} \qquad \text{(III.F.12)}$$

The resulting integrals are now in the form

$$\sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(k_o\gamma)}{\sqrt{2\pi}} \, f(k_o\gamma) \sqrt{\frac{2}{\pi\, k_o\kappa_{t\infty} r \sin\theta}} \; e^{ik_o\kappa_{t\infty} r \sin\theta} \; e^{ik_o\gamma\, r\cos\theta} \qquad \text{(III.F.13)}$$

where

$$f(k_o\gamma) = \begin{cases} -k_o^2(\gamma\kappa_{t\infty}\cos\theta + \kappa_{t\infty}^2\sin\theta) p_{\nu\infty} \; e^{-i\nu\frac{\pi}{2}} \; e^{-i\frac{\pi}{4}} \; e^{i\nu(\phi-\phi_s)} \; e^{-ik_o\gamma z_s} \\[2ex] \omega\mu_o\eta_\infty k_o\kappa_{t\infty} \, q_{\nu\infty} \; e^{-i\nu\frac{\pi}{2}} \; e^{-i\frac{\pi}{4}} \; e^{i\nu(\phi-\phi_s)} \; e^{-ik_o\gamma z_s} \end{cases} \qquad \text{(III.F.14)}$$

These integrals can be evaluated by the methods of saddle point integration if the integration is transformed to the complex α plane by means of the substitution

$$\gamma \equiv \kappa_\infty \cos\alpha \qquad \text{(III.F.15)}$$

This leads to the form

$$- \sum_{\nu=-\infty}^{\infty} \int_{\substack{\pi - i\infty \\ \Gamma}}^{i\infty} d\alpha \; f(k_o \kappa_\infty \cos \alpha) \, \frac{1}{\pi} \sqrt{\frac{k_o \kappa_{t\infty}}{r \sin \theta}} \; e^{ik_o \kappa_\infty r \cos(\alpha - \theta)} \qquad \text{(III.F.16)}$$

where the path of integration in the complex α plane is shown in Figure 16.

The next step is to transform the contour to the path of steepest descent. The path is defined by

$$\cos(\alpha - \theta) \equiv 1 + i\chi^2 \qquad \text{(III.F.17)}$$

where χ is to range from $-\infty$ to $+\infty$.

The integral then becomes

$$\sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} d\chi \sqrt{\frac{2}{i}} \, \frac{1}{\sqrt{1 + i\frac{\chi^2}{2}}} \, f\left\{k_o \kappa_\infty \cos[\cos^{-1}(1 + i\chi^2) + \theta]\right\}$$
$$\frac{1}{\pi} \sqrt{\frac{k_o \kappa_{t\infty}}{r \sin \theta}} \; e^{ik_o \kappa_\infty (1 + i\chi^2) r} \qquad \text{(III.F.18)}$$

The integrand can now be expanded in a power series in χ^2 and the integration performed term by term. This leads to

$$\sqrt{\frac{2}{i\pi}} \; \frac{e^{ik_o \kappa_\infty r}}{r} \sum_{\nu=-\infty}^{\infty} f(k_o \kappa_\infty \cos \theta) \qquad \text{(III.F.19)}$$

where the remaining terms contain higher powers of $1/r$.

By the use of the asymptotic expansion of the integrals, the theta and phi components of the field vectors become

$$E^{\theta} = i\sqrt{\frac{2}{\pi}}\, k_o^2 \kappa_\infty^2 \sin\theta\, \frac{e^{ik_o\kappa_\infty(r - z_s\cos\theta)}}{r} \sum_{\nu=-\infty}^{\infty} p_{\nu\infty}\Big|_{k_o\kappa_\infty\cos\theta} e^{-i\nu\frac{\pi}{2}} e^{i\nu(\phi-\phi_s)}$$

$$E^{\phi} = -i\sqrt{\frac{2}{\pi}}\, \omega\mu_o\eta_\infty k_o\kappa_\infty \sin\theta\, \frac{e^{ik_o\kappa_\infty(r - z_s\cos\theta)}}{r} \sum_{\nu=-\infty}^{\infty} q_{\nu\infty}\Big|_{k_o\kappa_\infty\cos\theta}$$

$$\times\; e^{-i\nu\frac{\pi}{2}} e^{i\nu(\phi-\phi_s)} \qquad \text{(III.F.20)}$$

and

$$H^{\theta} = -\frac{k_o\kappa_\infty}{\omega\mu_o\eta_\infty}\, E^{\phi}$$

$$H^{\phi} = \frac{k_o\kappa_\infty}{\omega\mu_o\eta_\infty}\, E^{\theta} \qquad \text{(III.F.21)}$$

Note that $p_{\nu\infty}$ and $q_{\nu\infty}$ are evaluated at the value of $k_o\kappa_\infty\cos\theta$.

G. Turnstile Antenna

1. Description

A turnstile antenna consists of two mutually perpendicular dipole antennas, one fed 90° out of phase with respect to the other. For right circular polarized radiation in the z direction the dipoles are oriented in the x and y directions, with the y oriented dipole fed -90° out of phase with respect to the x oriented dipole.

If the antenna is in free space and located at the origin of a spherical coordinate system, then in the far field of the antenna the theta and phi components of the electric field vector are (13)

$$E^\theta = i\omega\mu_o \, i_s l_s \frac{e^{ik_o r}}{4\pi r} e^{i\phi} \cos\theta$$

$$E^\phi = -\omega\mu_o \, i_s l_s \frac{e^{ik_o r}}{4\pi r} e^{i\phi} \qquad \text{(III.G.1)}$$

If the antenna is placed on the z axis $\lambda_o/4$ above an infinite ground plane located in the xy plane at $z = 0$, then the effects of the ground plane are obtained by multiplying Eq. III.G.1 by the array factor (13)

$$-i2 \sin\left(\frac{\pi}{2}\cos\theta\right) \qquad \text{(III.G.2)}$$

The resulting left and right circular polarized components of the radiation are

$$E^l = -2^{3/2} \omega\mu_o i_s l_s \frac{e^{ik_o r}}{4\pi r} e^{i\phi} \sin^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi}{2}\cos\theta\right) \equiv i\omega\mu_o \frac{e^{ik_o r}}{4\pi r} l(\theta,\phi)$$

$$E^r = 2^{3/2}\,\omega\mu_o i_s l_s \frac{e^{ik_o r}}{4\pi r} e^{i\phi}\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi}{2}\cos\theta\right) \equiv i\omega\mu_o \frac{e^{ik_o r}}{4\pi r} r(\theta,\phi) \qquad \text{(III.G.3)}$$

As indicated, l and r are functions of the angular coordinates only.

The point source character of the antenna becomes evident on considering the power radiated into the far field. The Poynting vector is

$$\underline{s} = \hat{r}\,\frac{1}{8}\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{1}{\lambda_o^2}\,\frac{1}{r^2}\,a(\theta,\phi) \qquad \text{(III.G.4)}$$

where

$$a(\theta,\phi) \equiv |l(\theta,\phi)|^2 + |r(\theta,\phi)|^2 \qquad \text{(III.G.5)}$$

In discussing the power flow in the far field it is convenient to use, instead of the Poynting vector, the intensity $I(\theta,\phi)$ which is the power radiated per solid angle in the radial direction. The intensity is defined as

$$I(\theta,\phi) \equiv r^2|\underline{s}| = \frac{1}{8}\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{1}{\lambda_o^2}\,a(\theta,\phi) \qquad \text{(III.G.6)}$$

which is independent of the radial distance.

The power distribution in the far field is conveniently specified in terms of a gain function $g(\theta,\phi)$ with respect to an isotropic radiator. The gain is defined as

$$g(\theta,\phi) \equiv \frac{I(\theta,\phi)}{\dfrac{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, I(\theta,\phi)}{4\pi}} \qquad \text{(III.G.7)}$$

The gain of the antenna in free space is shown in Figure 17.

The gain of the antenna in the presence of an infinitely long cylindrical plasma shell is obtained from the theta and phi components of the electric field vector as given by Eq. III.F.20. The resulting radiation patterns in the presence of the plasma will be compared with the free space patterns.

2. Numerical Results

Numerical computations have been carried out for the special case in which the antenna is located on the axis of an infinitely long cylindrical shell containing a uniform, lossless, and isotropic plasma. The antenna is assumed to operate at the signal frequency of 400 MHz, and the shell is assumed to extend from 0.75 m to 1.25 m, which corresponds, respectively, to $k_o\rho_o \simeq 6$ and $k_o\rho_\infty \simeq 10$ at 400 MHz. At this signal frequency the thickness of the shell is smaller than the wavelength in free space.

This plasma configuration adequately describes the near wake of a capsule entering the Martian atmosphere with a step distribution in the electron concentration of the plasma. A more accurate description of the electron concentration is not expected to significantly alter the resulting radiation patterns of the antenna. Since the antenna is located on-axis, the radiation patterns are symmetric about the axis of the shell; and therefore, only the theta dependence need be considered in the work that follows. For this case only the $\nu = \pm 1$ terms need to be summed in Eq. III.F.20. The $\nu = \pm 1$ terms give rise to the dipolar modes as discussed in the references (17). The effects of locating the antenna off-axis and the effects of varying the

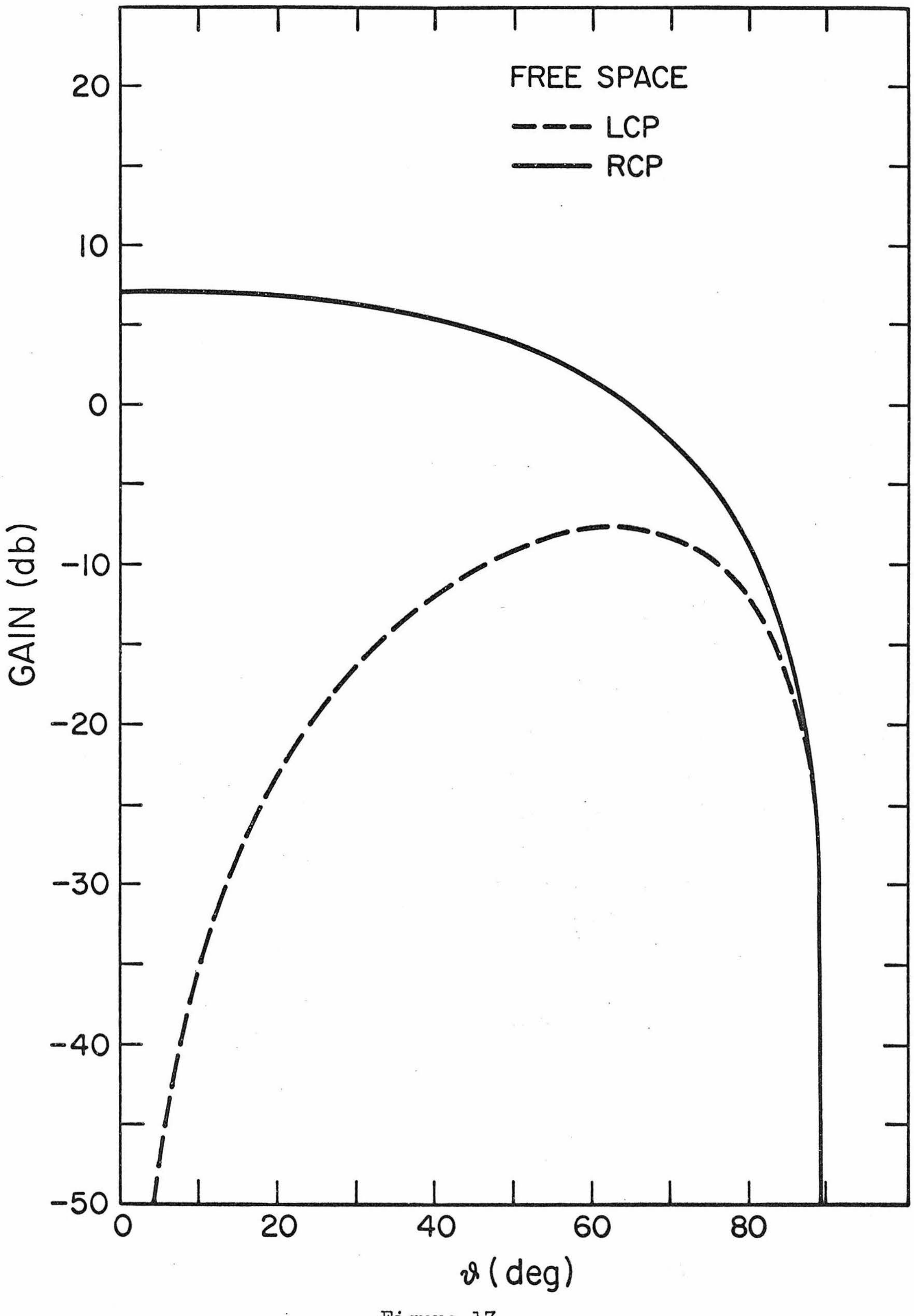

FREE SPACE
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 17

shell radius are also discussed in the references (18).

The gain of the antenna is now investigated as a function of theta with the electron concentration of the plasma and the velocity of the plasma relative to the antenna as parameters. For the low velocity cases, the results of these investigations correspond to an entry into the Martian atmosphere.

At the operating frequency of 400 MHz, the plasma is transparent for electron concentrations below 1.99×10^9 e^-/cc. Above this electron concentration, the plasma is opaque and the condition of blackout exists. Therefore, numerical data were taken for a range of electron concentrations below 1.99×10^9 e^-/cc. In each case the gain of the antenna was developed as a function of theta for various velocities of the plasma. The resulting radiation patterns are shown in Figures 18 through 26.

As is evident from an examination of these figures, the effects of the plasma begin to appear at $1 \times 10^7 e^-$/cc. Below this value of the electron concentration, the interaction of the plasma with the radiation of the antenna is too weak to cause a noticeable deviation in the values of the gain function from the free space values. Above this value of the electron concentration, however, the plasma has a noticeable effect on the radiation of the antenna. In particular, the presence of the plasma causes a null to appear in the radiation patterns for small values of theta. Also, for larger values of theta, the values of the gain function oscillate about the free space values.

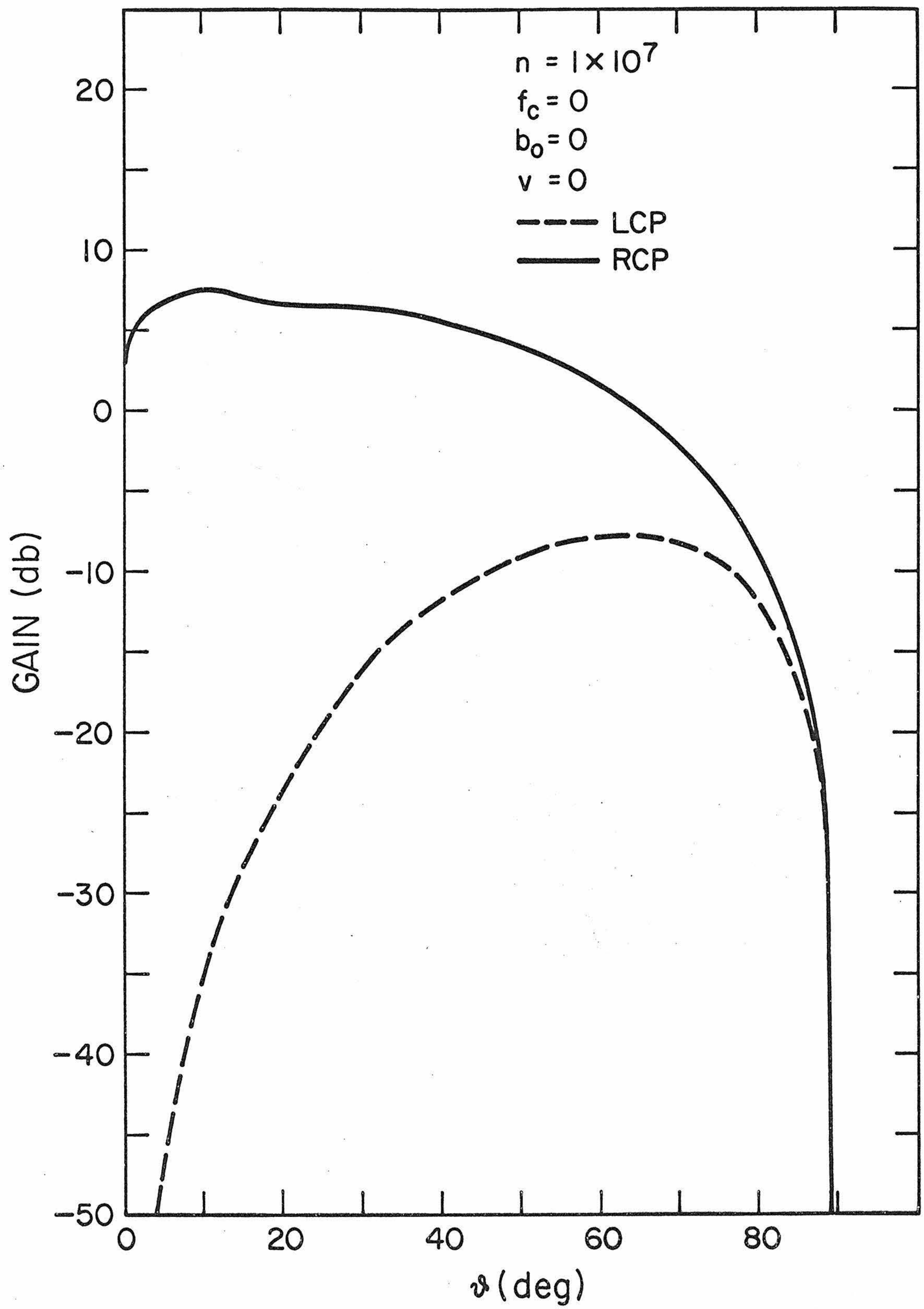
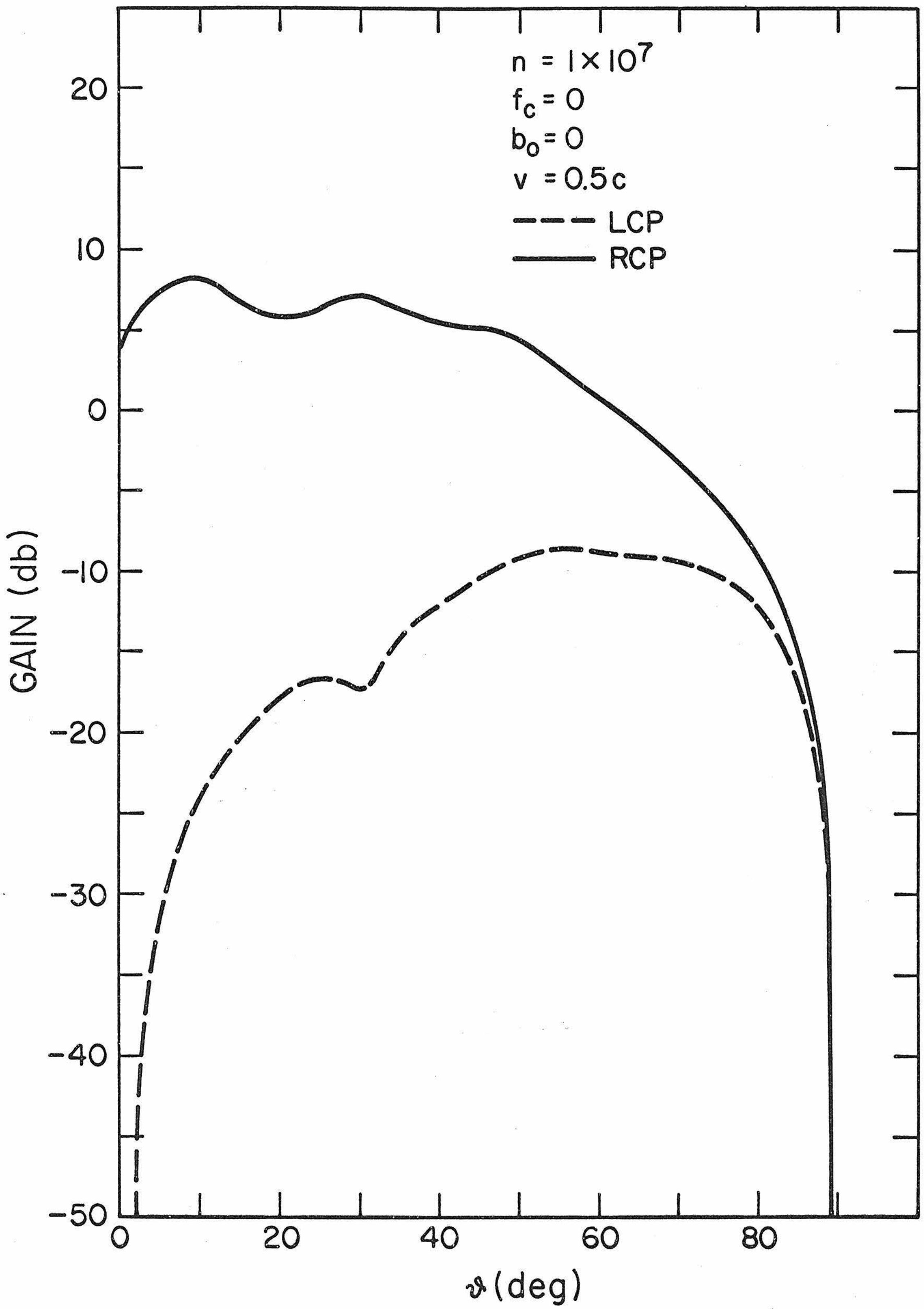
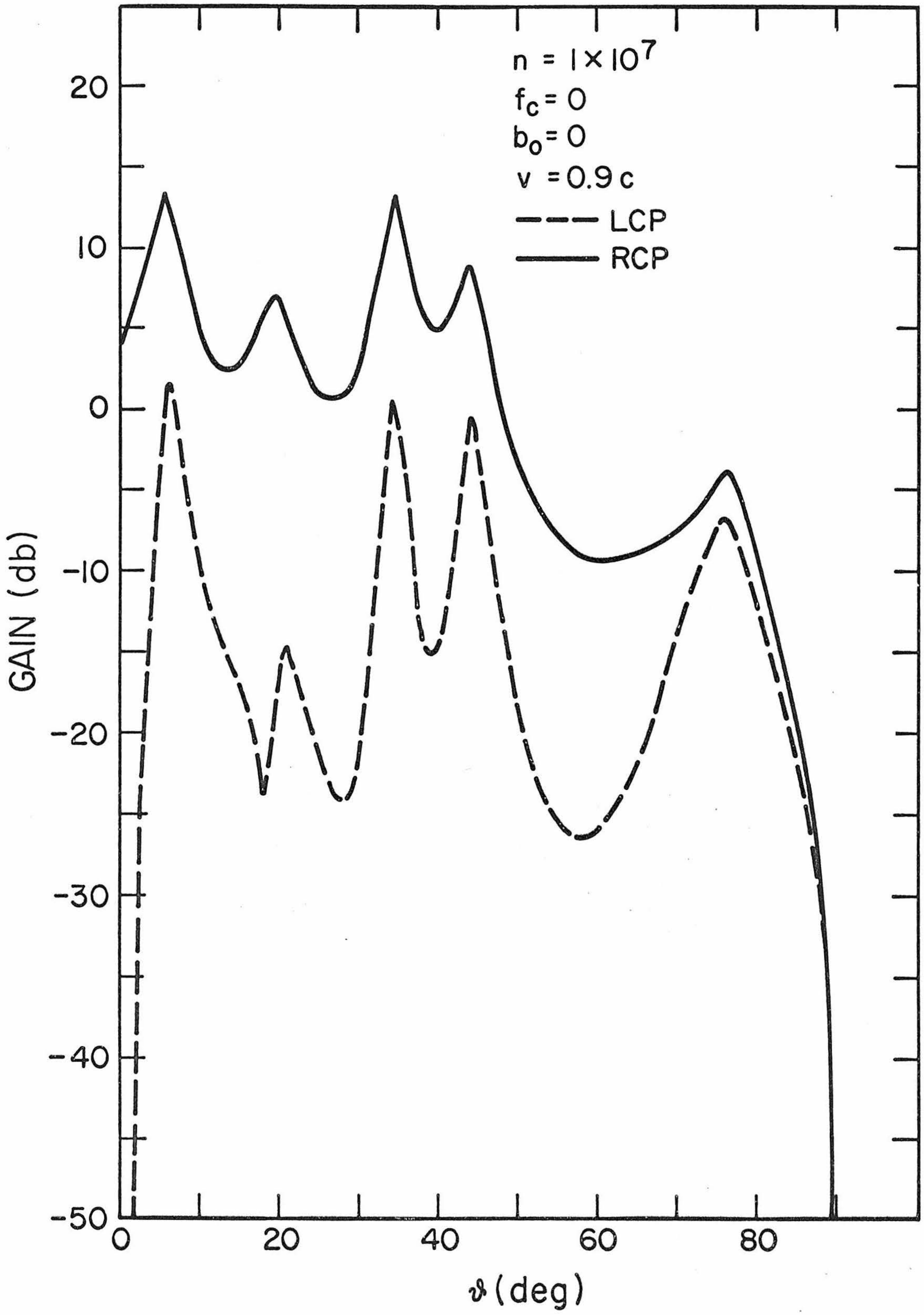

n = 1×10^7
f_c = 0
b_o = 0
v = 0
LCP
RCP
GAIN (db)
20
10
0
-10
-20
-30
-40
-50
0
20
40
60
80
ϑ (deg)

Figure 18



n = 1×10^7
f_c = 0
b_o = 0
v = 0.5c
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 19



n = 1×10^7
f_c = 0
b_o = 0
v = 0.9 c
LCP
RCP
GAIN (db)
20
10
0
-10
-20
-30
-40
-50
0
20
40
60
80
ϑ (deg)

Figure 20

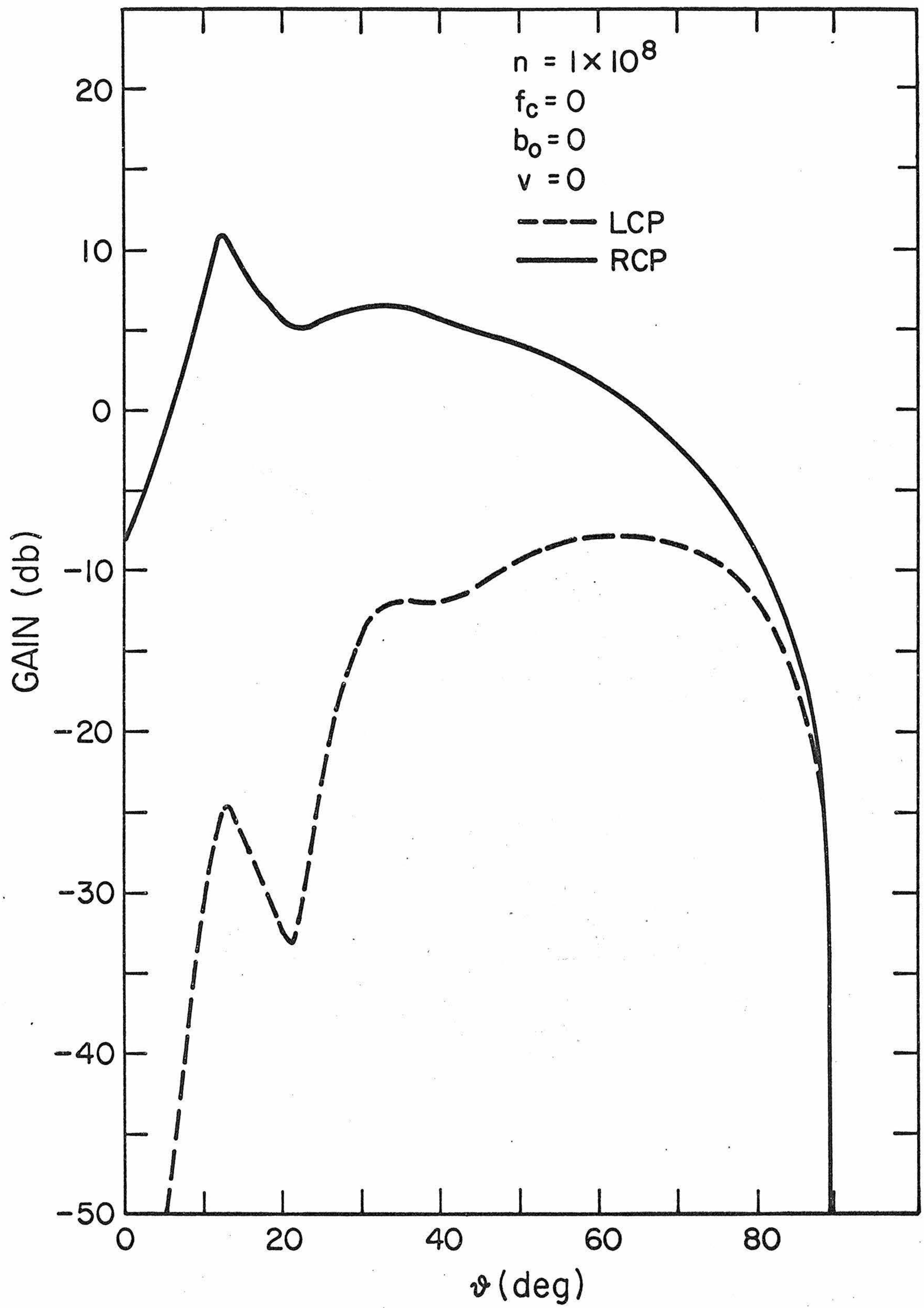
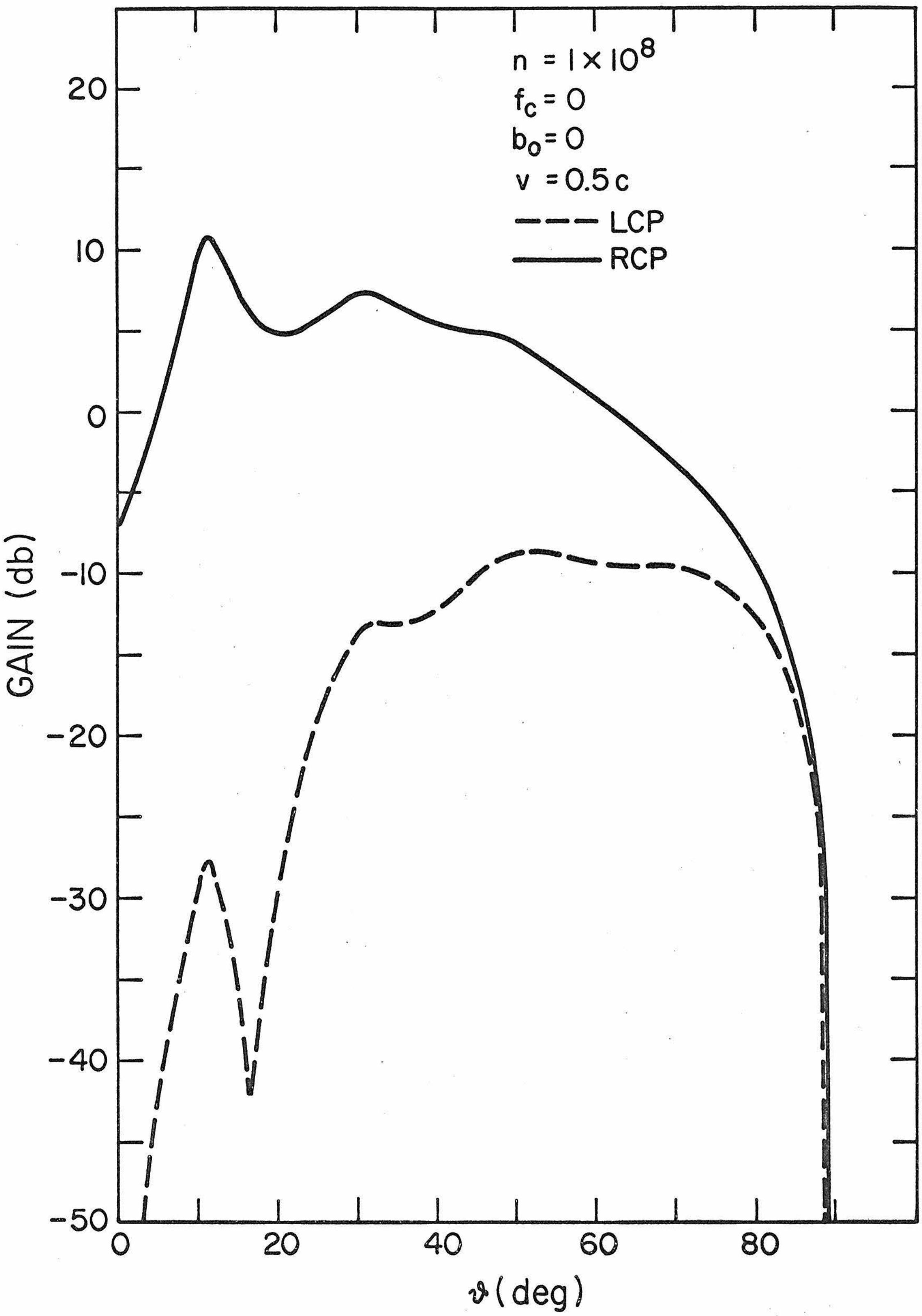
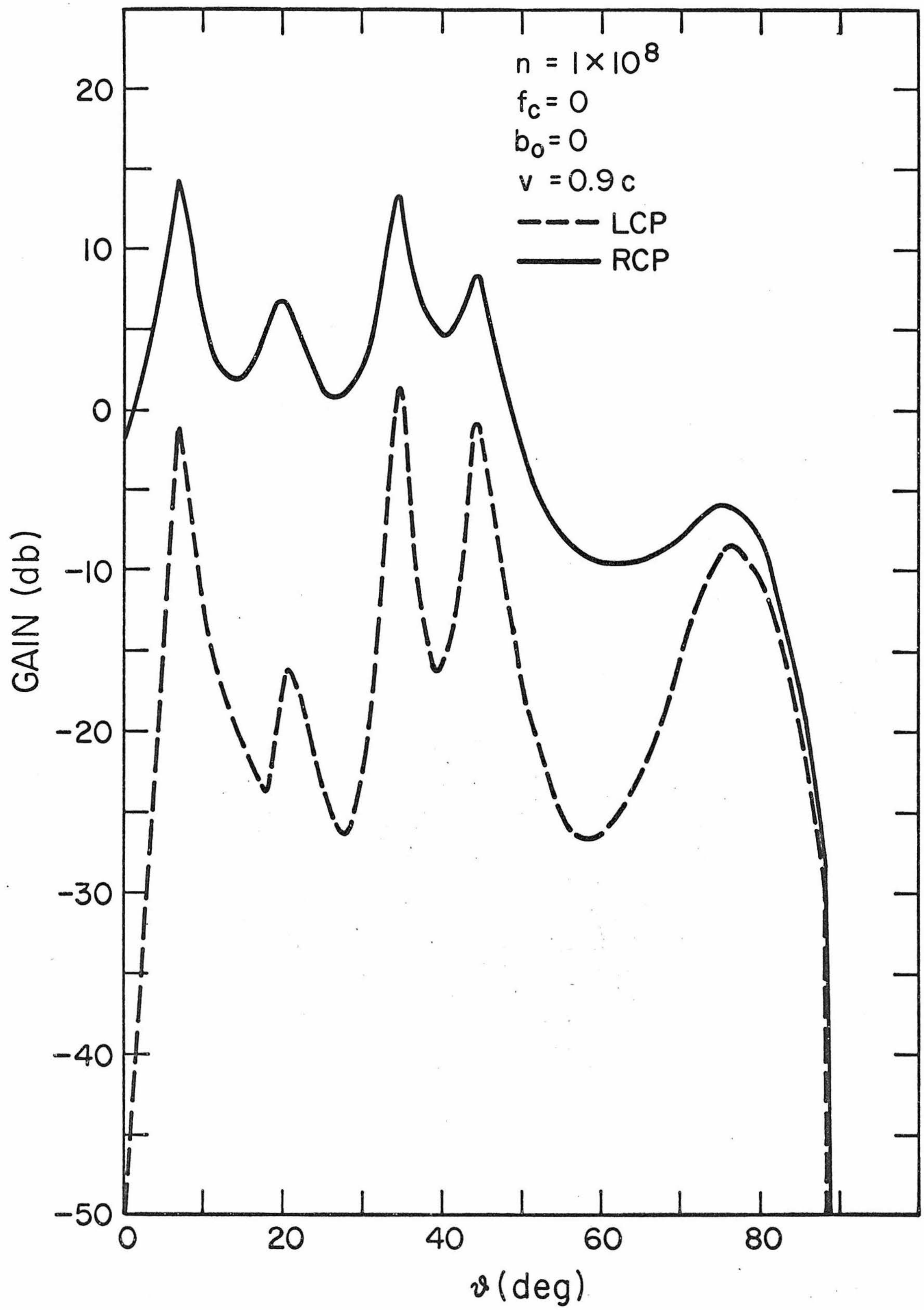

$n = 1 \times 10^8$
$f_c = 0$
$b_o = 0$
$v = 0$
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 21



n = 1 × 10^8
f_c = 0
b_o = 0
v = 0.5 c
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 22



$n = 1 \times 10^8$
$f_c = 0$
$b_o = 0$
$v = 0.9c$
LCP
RCP
GAIN (db)
20
10
0
-10
-20
-30
-40
-50
0
20
40
60
80
ϑ (deg)

Figure 23

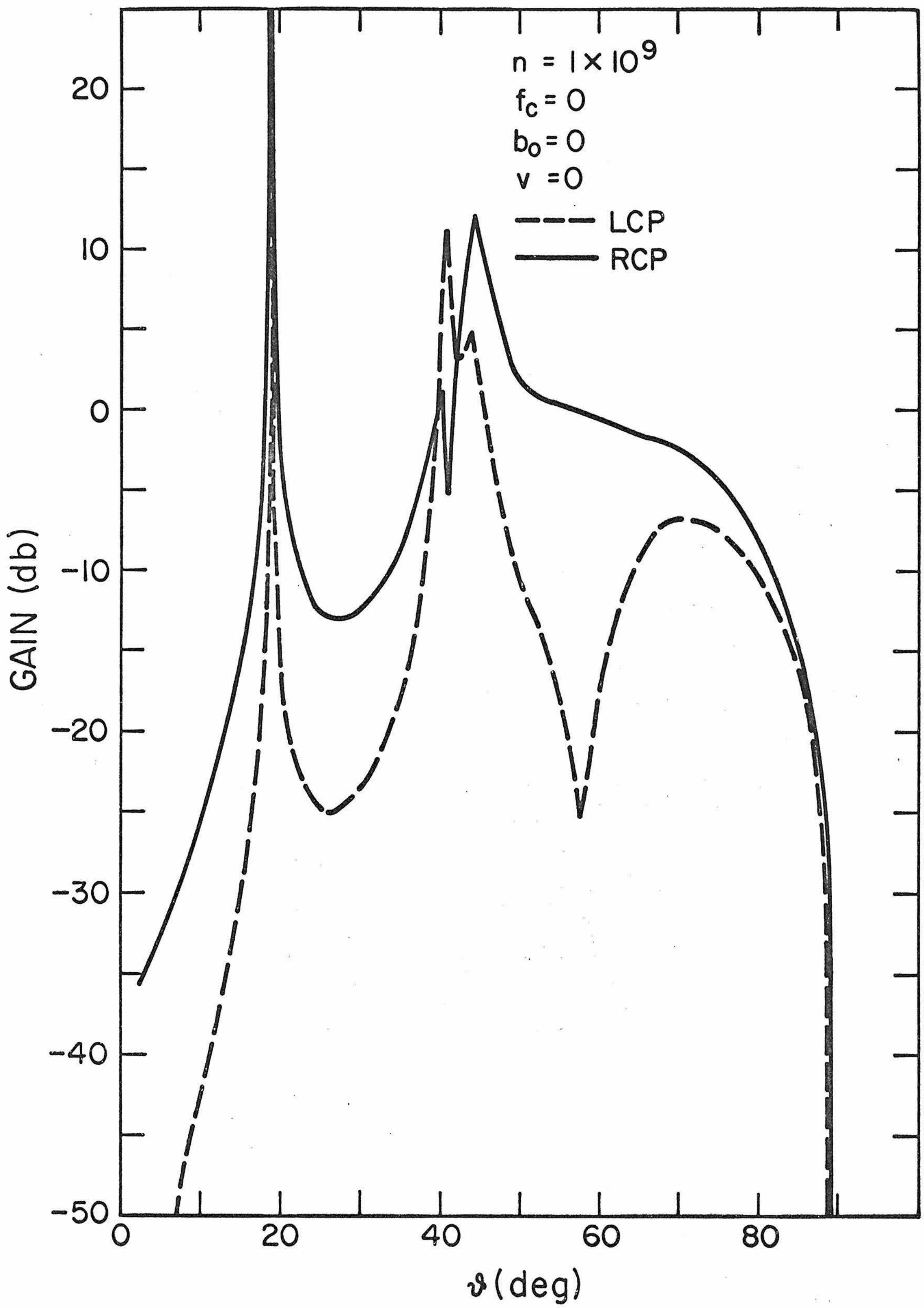

n = 1×10^9
fc = 0
bo = 0
v = 0
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 24

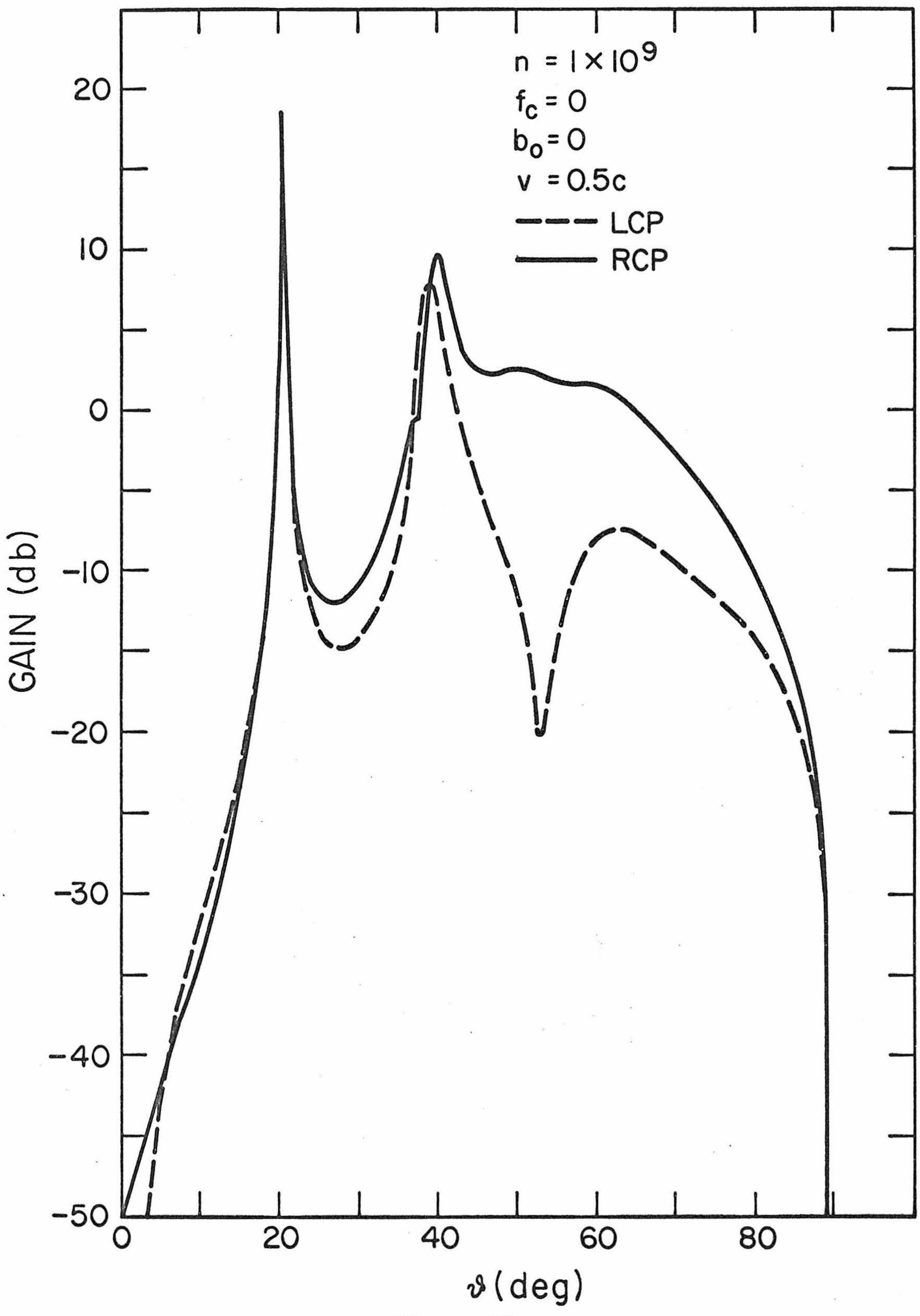
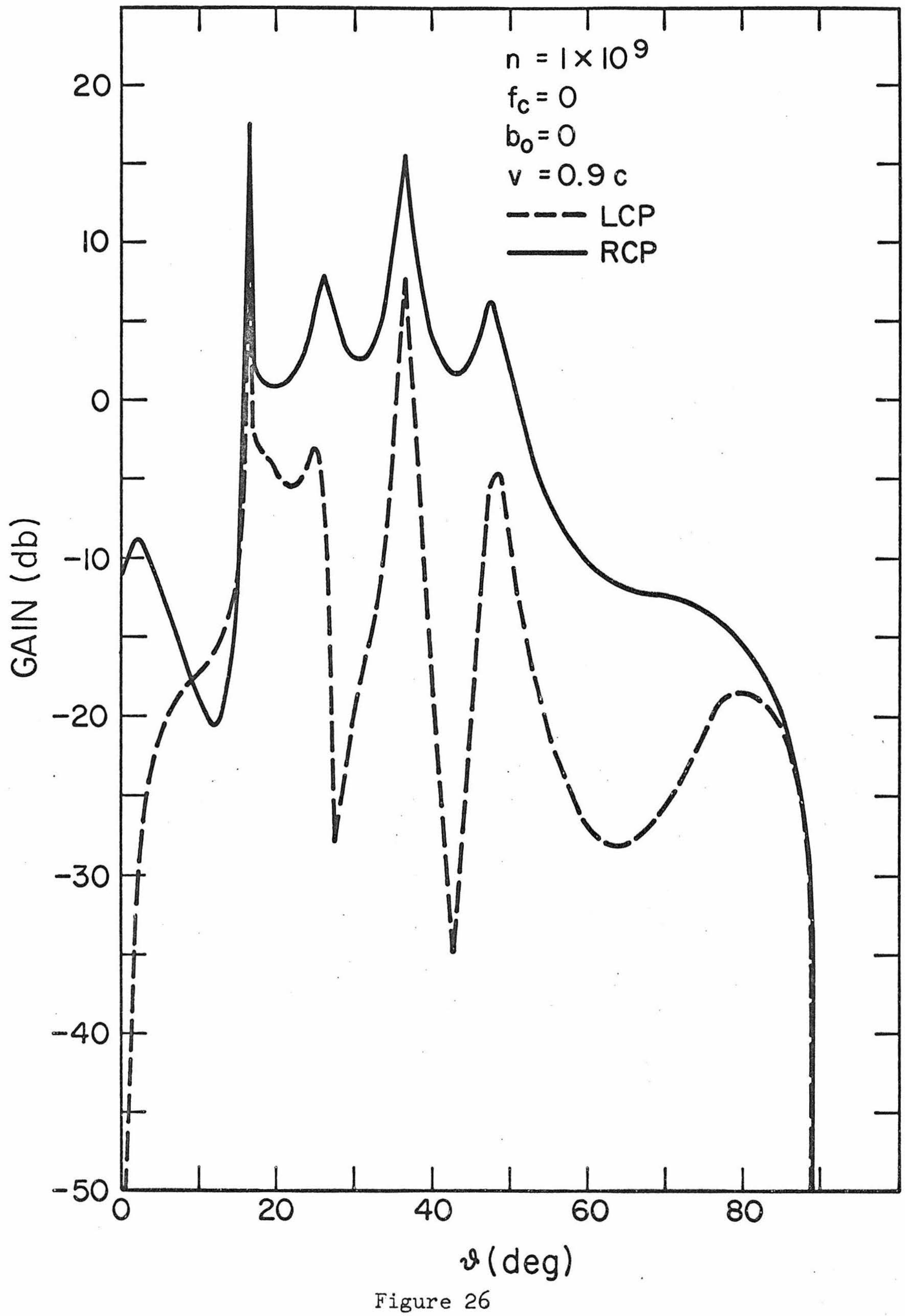

n = 1×10^9
f_c = 0
b_o = 0
v = 0.5c
LCP
RCP
20
10
0
-10
-20
-30
-40
-50
GAIN (db)
0
20
40
60
80
ϑ (deg)

Figure 25



n = 1 × 10^9
f_c = 0
b_o = 0
v = 0.9 c
LCP
RCP
GAIN (db)
20
10
0
-10
-20
-30
-40
-50
0
20
40
60
80
ϑ (deg)

Figure 26

As the electron concentration is increased above $1 \times 10^7 e^-/cc$, the extent of the null region increases and the peaks become more numerous and more sharply defined. As the electron concentration approaches the cutoff value of $1.99 \times 10^7 e^-/cc$, very sharp peaks also occur within the null regions.

The presence of the null region can be explained from physical considerations with the principles of geometrical optics. Since the radiation from the antenna is refracted by the free space, plasma interface at the inner radius of the shell, there exists a critical angle which corresponds to total internal reflection into the free space region. This critical angle as determined from Snell's law of refraction is

$$\theta_c = \cos^{-1} \sqrt{\varepsilon_r} \tag{III.G.8}$$

Therefore,

$$\theta_c = 3.6^\circ \quad \text{at} \quad 1 \times 10^7 e^-/cc$$
$$\theta_c = 13.1^\circ \quad 1 \times 10^8 e^-/cc$$
$$\theta_c = 45.2^\circ \quad 1 \times 10^9 e^-/cc$$

For radiation incident on the inner radius of the shell at angles $\theta \leq \theta_c$, the radiation is propagated parallel to the surface of the shell. This surface wave travels at a speed less than the speed of light in vacuum and can produce radiation only at a discontinuity in the guiding surface. Therefore, the radiation of the antenna emitted into the angles $\theta \leq \theta_c$ cannot reach the free space region outside of the shell, and thus the values of the gain function are reduced for these angles.

The oscillations in the values of the gain function about the free space values for angles $\theta > \theta_c$ are the result of standing waves and plasma resonances supported by the plasma shell. As the density of the plasma is increased, more peaks occur in the radiation patterns, since the plasma is capable of supporting higher order standing waves as it becomes denser.

The sharp peaks in the null region $\theta \leq \theta_c$ are caused by the waves propagating along the outer radius of the shell. These waves travel with a speed greater than the speed of light in vacuum and are attenuated as they travel, indicating a continuous leakage of energy out of the shell and into the free space region. Waves possessing these characteristics are distinguished as leaky waves and have been observed in other studies (19).

Since the leaky waves decay exponentially in the radial direction inside the plasma shell, the leakage of energy across the outer radius of the shell is small at any one point on the shell; however, the radiation interferes constructively at the same angle since the wave number of the leaky wave in the plasma shell is constant and results in a sharp peak in the radiation patterns. This leaky wave angle is

$$\theta_1 = \cos^{-1} \frac{\beta}{k_o} \qquad \text{(III.G.9)}$$

where β is the propagation constant of the leaky wave.

As the velocity of the plasma is increased from zero to values near the speed of light in vacuum, the peaks of the stationary patterns are shifted to smaller angles in theta. The amount of the shift in any given case is proportional to the velocity of the plasma. This is the effect one would expect that a moving medium would have on the radiation of an antenna. The effect of shifting the peaks of the radiation patterns to smaller angles in theta tends to decrease the severity of the on-axis null region and tends to spread out the peaks so that they are no longer as sharp or as intense.

Note, however, that a more interesting effect also occurs. As the plasma moves, more peaks are introduced into the radiation patterns. This effect is due to the Lorentz-contraction observed along the axis of the cylinder. As the plasma moves, it appears to increase in density and more peaks are observed in the radiation patterns in the rest frame of the antenna, in addition to the shifting of the peaks to smaller angles in theta.

From an examination of the various radiation patterns, it is clear that no serious motional effects or depolarization effects occur during the entry of the capsule into the Martian atmosphere and communications with the capsule can be satisfactorily carried out.

IV. CONCLUSION

The interaction between the ionized wake of a capsule entering the Martian atmosphere and the circularly polarized radiation emitted by a turnstile antenna located on the aft part of the capsule have been investigated in this study.

It has been shown that blackout occurs during the entry of a capsule into the Martian atmosphere, and that the calculated duration of the blackout depends on the signal frequency of the antenna and the mathematical model chosen to represent the Martian atmosphere.

The gain of the transmitting antenna has also been studied before and after blackout. The results of the numerical computations carried out at 400 MHz show that the effects of the plasma on the radiation from the antenna begin to appear at the electron concentration of $1 \times 10^7 e^-/cc$. In particular, the radiation patterns of the antenna develop a null region for small values of theta. Also, sharp peaks occur within the null region of the patterns for values of the electron concentration approaching the cutoff value of $1.99 \times 10^9 e^-/cc$. For larger values of theta, the values of the gain function of the antenna oscillate about the free space values. The effects of the plasma on the radiation emitted by the antenna are seen to depend on the electron concentration of the plasma and the velocity of the plasma relative to the antenna.

It is concluded that, for the low velocity case corresponding to an entry into the Martian atmosphere, no serious motional or depolarization effects occur, and that communications with the capsule can be satisfactorily carried out when the condition of blackout does not exist.

APPENDIX

Projection Operators

In what follows let the operator which projects any vector onto itself be denoted by $\underline{\underline{u}}$, i.e.,

$$\underline{v} \equiv \underline{\underline{u}} \circ \underline{v} \tag{A.1}$$

Also, let the dual of any vector $\underline{v}$ be denoted by $*\underline{v}$, i.e.,

$$(*\underline{v})^{jk} \equiv \sum_{l} \varepsilon^{jkl} v^{l} \tag{A.2}$$

where

$$\varepsilon^{jkl} \equiv \begin{cases} +1 \text{ if } jkl \text{ forms an even permutation of } 123 \\ -1 \text{ if } jkl \text{ forms an odd permutation of } 123 \\ 0 \text{ otherwise} \end{cases} \tag{A.3}$$

If the projection of the vector $\underline{v}$ on the plane normal to the z direction is denoted by $\underline{v}^{t}$, then

$$\underline{v}^{t} = \underline{\underline{t}} \cdot \underline{v} \tag{A.4}$$

where the projection operator $\underline{\underline{t}}$ is defined as

$$\underline{\underline{t}} \equiv \underline{\underline{u}} - \hat{z}\hat{z} \tag{A.5}$$

If the projection of the vector $\underline{v}$ on the plane normal to the z direction, followed by a rotation through θ radians about the z direction, is denoted by $\underline{v}^{s}$, then

$$v^{s} = \underline{\underline{r}}_{\theta} \circ \underline{\underline{t}} \cdot \underline{v} \tag{A.6}$$

where the rotation operator $\underline{\underline{r}}_\theta$ is defined as

$$\underline{\underline{r}}_\theta \equiv \underline{\underline{u}} \cos \theta + \hat{z}\hat{z}(1 - \cos \theta) - *\hat{z} \sin \theta \qquad (A.7)$$

For the special case of $\theta = \pi/2$, let the projection operator $\underline{\underline{s}}$ be defined as

$$\underline{\underline{s}} \equiv \underline{\underline{r}}_{\pi/2} \cdot \underline{\underline{t}} \qquad (A.8)$$

then

$$\underline{v}^s = \underline{\underline{s}} \cdot \underline{v} \qquad (A.9)$$

The following identities among the projection operators are evident:

$$\begin{aligned}
(\underline{v}^s)^s &= - \underline{v}^t \\
\underline{v}^s \cdot \underline{v}^t &= 0 \\
\underline{u}^s \cdot \underline{v}^t &= - \underline{u}^t \cdot \underline{v}^s \\
\underline{u}^s \cdot \underline{v}^s &= \underline{u}^t \cdot \underline{v}^t \\
(\underline{u} \wedge \underline{v})^z &= \underline{u}^s \cdot \underline{v}^t \\
(\underline{u} \wedge \underline{v})^t &= u^z \underline{v}^s - \underline{u}^s v^z
\end{aligned} \qquad (A.10)$$

If the dyad $\underline{\underline{c}}$ is introduced by letting

$$\underline{\underline{c}} \equiv - *\hat{z} \qquad (A.11)$$

then $\underline{v}^s$ and $\underline{v}^t$ are related by

$$\underline{v}^s = \underline{\underline{c}} \cdot \underline{v}^t \qquad (A.12)$$

or

$$\underline{v}^s = \hat{z} \wedge \underline{v}^t \tag{A.13}$$

Also, since the inverse of $\underline{\underline{c}}$ is just $-\underline{\underline{c}}$,

$$\underline{\underline{c}} \cdot \underline{\underline{c}} = - \underline{\underline{t}} \tag{A.14}$$

In terms of the dyad $\underline{\underline{c}}$, the identities A.10 become

$$\begin{aligned}
&\underline{\underline{c}} \cdot \underline{\underline{c}} \cdot \underline{v}^t = -\underline{v}^t \\
&\underline{v}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t = 0 \\
&\underline{u}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t = -\underline{v}^t \cdot \underline{\underline{c}} \cdot \underline{u}^t \\
&\underline{\underline{c}} \cdot \underline{u}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t = \underline{u}^t \cdot \underline{v}^t \\
&(\underline{u} \wedge \underline{v})^z = -\underline{u}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t \\
&(\underline{u} \wedge \underline{v})^t = u^z \underline{\underline{c}} \cdot \underline{v}^t - v^z \underline{\underline{c}} \cdot \underline{u}^t
\end{aligned} \tag{A.15}$$

If a similar notation is used for the differential operator $\underline{\nabla}$, i.e.,

$$\underline{\nabla}^t = \underline{\underline{t}} \cdot \underline{\nabla} \tag{A.16}$$

$$\underline{\nabla}^s = \underline{\underline{s}} \cdot \underline{\nabla} \tag{A.17}$$

then

$$\underline{\nabla} \wedge \underline{v} \equiv -\hat{z}\ \underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t - \underline{\underline{c}} \cdot (\underline{\nabla}^t v^z + \nabla^z \underline{v}^t) \tag{A.18}$$

and

$$\begin{aligned}
\underline{\nabla} \wedge \underline{\nabla} \wedge \underline{v} \equiv & -\hat{z}[(\nabla^t)^2\, v^z + \nabla^z \underline{\nabla}^t \cdot \underline{v}^t] \\
& + \underline{\underline{c}} \cdot \underline{\nabla}^t (\underline{\nabla}^t \cdot \underline{\underline{c}} \cdot \underline{v}^t) - \nabla^z \underline{\nabla}^t v^z - (\nabla^z)^2\, \underline{v}^t
\end{aligned} \tag{A.19}$$

REFERENCES

1. L. D. Kaplan, D. Munch and H. Spinrad, "An Analysis of the Spectrum of Mars," The Astrophysical Jour. 139, 1 (1964).

2. A. Kliore, D. L. Cain, G. S. Levy, V. R. Eshleman, G. Fjeldbo, and F. D. Drake, "Occultation Experiment: Results of the First Direct Measurement of Mars's Atmosphere and Ionosphere," Science 149, 1243 (1964).

3. A. Kliore, D. L. Cain, T. W. Hamilton, "Determination of Some Physical Properties of the Atmosphere of Mars from Changes in the Doppler Signal of a Spacecraft on an Earth-Occultation Trajectory," Tech. Report No. 32-674, Jet Propulsion Laboratory, Caltech, Oct. 15, 1964.

4. G. Fjeldbo, W. C. Fjeldbo and V. R. Eshleman, "Models for the Atmosphere of Mars Based on the Mariner 4 Occultation Experiment," Jour. Geophys. Res. 71, 2307 (1966).

5. W. J. Humphreys, Physics of the Air, Dover Publishers, Inc. 1964.

6. E. K. Casani, "Mars '71 Technical Study," Engineering Planning Document No. 427, Jet Propulsion Laboratory, Caltech, Aug. 15, 1966.

7. A. Ambrosio, "A General Atmospheric Entry Function and Its Characteristics," ARS Journal 32, 906 (1962).

8. D. F. Spencer, "An Evaluation of the Communication Blackout Problem for a Blunt Mars-Entry Capsule and a Potential Method for the Elimination of Blackout," Tech. Report No. 32-594, Jet Propulsion Laboratory, Caltech, April 15, 1964.

9. L. D. Landau and E. M. Lifshitz, Statistical Physics, Vol. 5, Course of Theoretical Physics, Addison-Wesley Publishing Co., Inc., 1958.

10. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Vol. 6, Course of Theoretical Physics, Addison-Wesley Publishing Co., Inc. 1959.

11. T. E. Horton, "The JPL Thermochemistry and Normal Shock Computer Program," Tech. Report No. 32-660, Jet Propulsion Laboratory, Caltech, Nov. 1, 1964.

12. J. D. Norgard, "A Martian Entry Propagation Study," to be published as Tech. Report No. 32-1393, Jet Propulsion Laboratory, Caltech, June 1, 1969.

13. C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw-Hill Book Company, 1965.

14. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 2nd Edition, 1944.

15. C. Moller, The Theory of Relativity, Oxford University Press, 1952.

16. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1965.

17. G. L. Yip and S. R. Seshardi, "Radiation from an Electric Dipole in an Axially Magnetized Plasma Column--Dipolar Modes," Canadian Jour. Phys. 45, 3627 (1967).

18. R. Woo and A. Ishimaru, "Radiation from a Circularly Polarized Antenna through the Ionized Wake of a Mars-Entry Capsule," to be published in IEEE Trans. Antennas and Propagation, July 1969.

19. L. O. Goldstone and A. A. Oliner, "Leaky-Wave Antennas I: Rectangular Waveguides," IRE Trans. Antennas and Propagation AP-7, 307 (1959).